ORF 307
Network Flows: Algorithms

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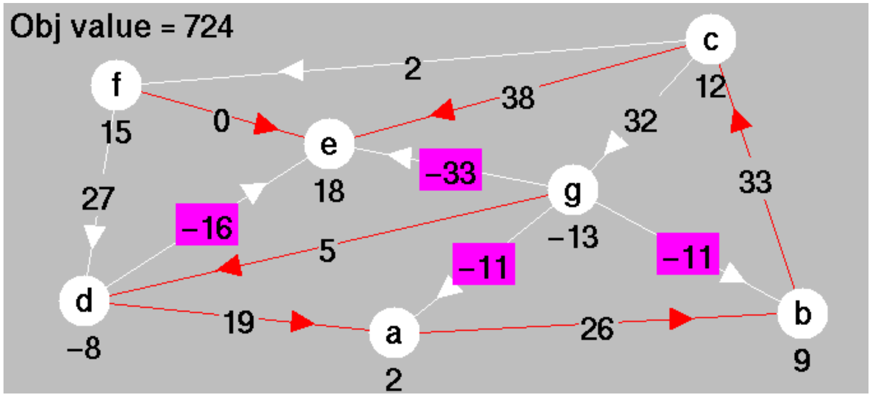
Agenda

- Primal Network Simplex Method
- Dual Network Simplex Method
- Two-Phase Network Simplex Method
- One-Phase Primal-Dual Network Simplex Method
- Planar Graphs
- Integrality Theorem

Primal Network Simplex Method

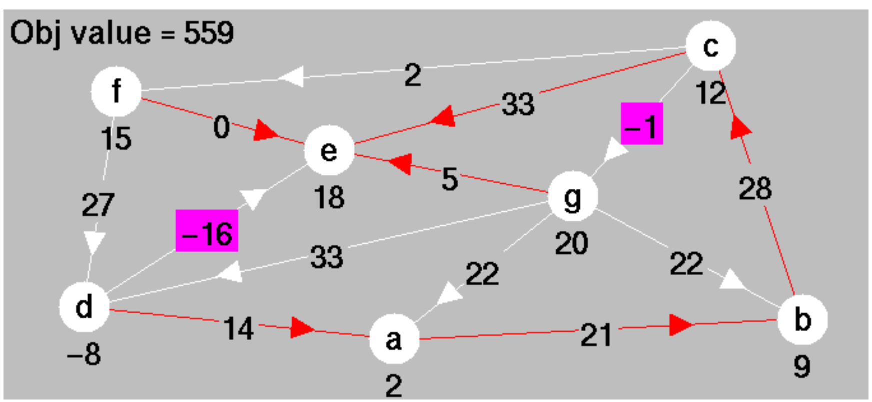
Used when all primal flows are nonnegative (i.e., primal feasible).



Pivot Rules:

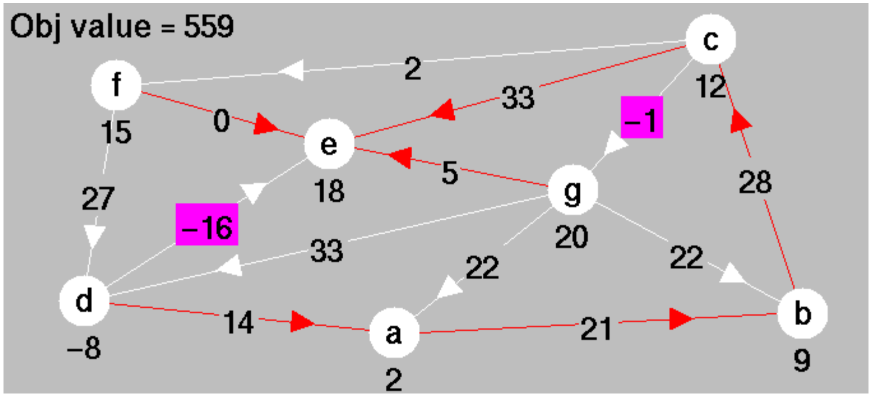
Entering arc: Pick a nontree arc having a negative (i.e. infeasible) dual slack.

Entering arc: (g,e)
Leaving arc: (g,d)

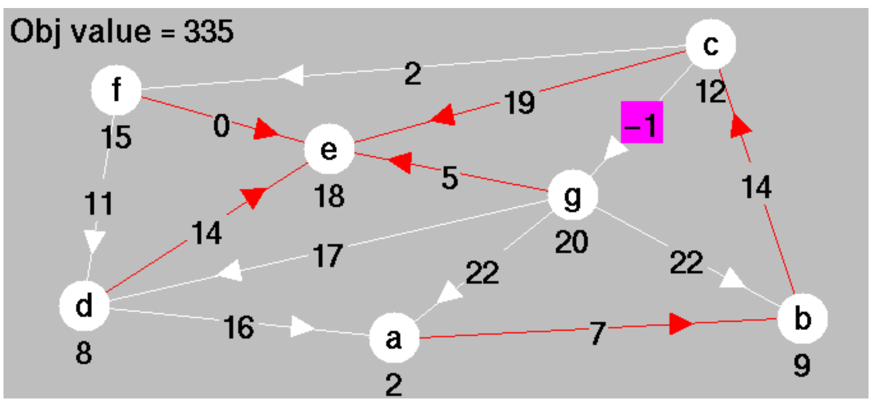


Leaving arc: Add entering arc to make a *cycle*.
Leaving arc is an arc on the cycle, pointing in the *opposite* direction to the entering arc, and
of all such arcs, it is the one with the *smallest* primal flow.

Primal Method—Second Pivot



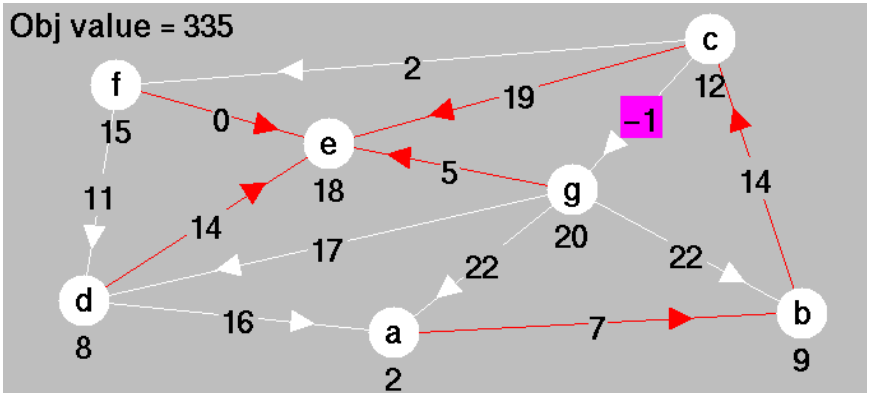
Entering arc: (d,e)
Leaving arc: (d,a)



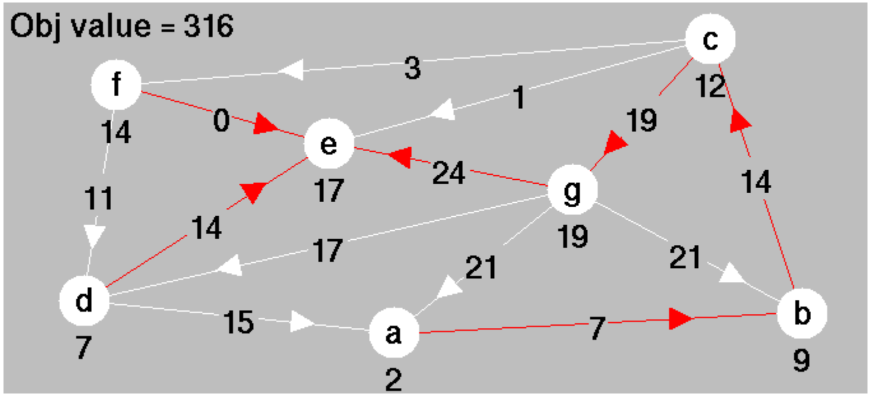
Explanation of leaving arc rule:

- Increase flow on (d,e).
- Each unit increase produces a unit *increase* on arcs pointing in the *same* direction.
- Each unit increase produces a unit *decrease* on arcs pointing in the *opposite* direction.
- The first to reach zero will be the one pointing in the opposite direction and having the smallest flow among all such arcs.

Primal Method—Third Pivot



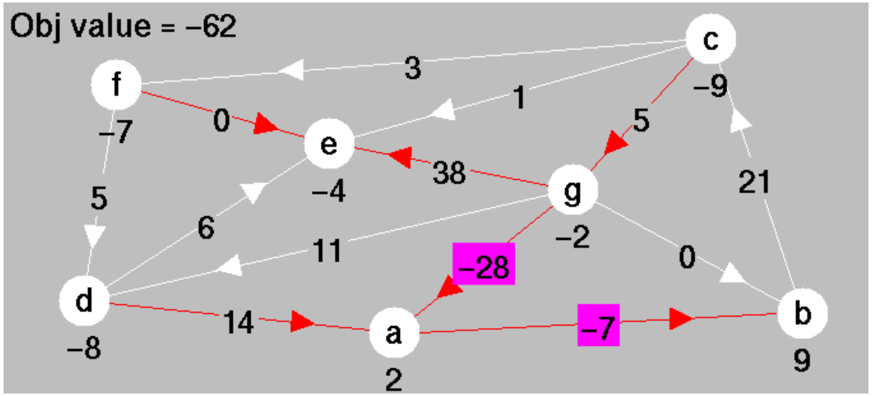
Entering arc: (c,g)
Leaving arc: (c,e)



Optimal!

Dual Network Simplex Method

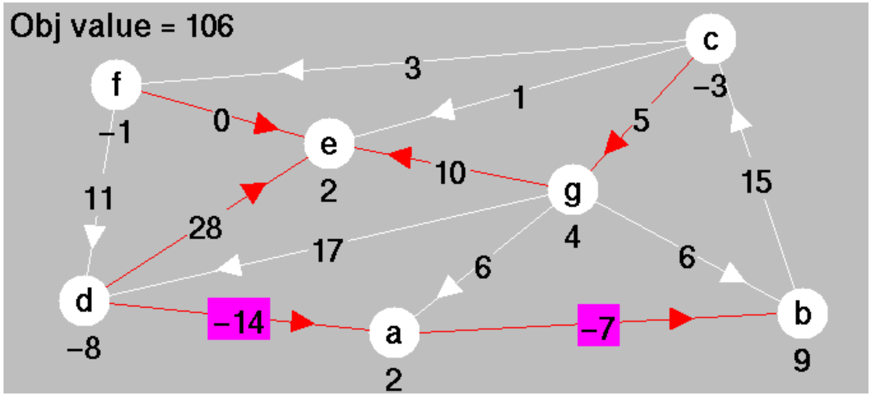
Used when all dual slacks are nonnegative (i.e., dual feasible).



Pivot Rules:

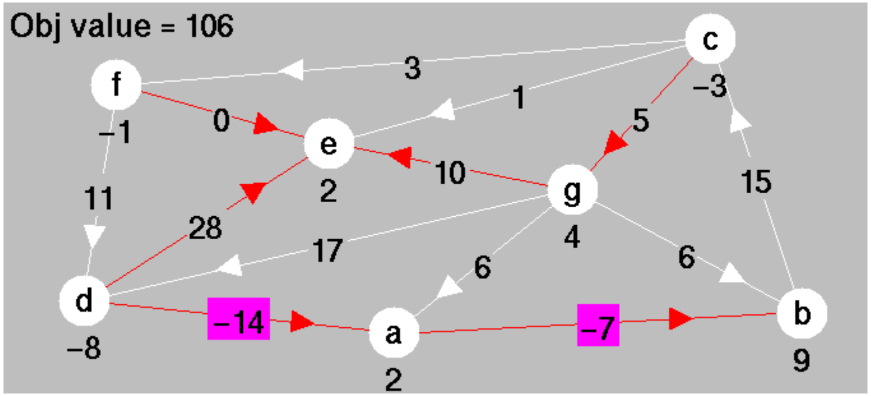
Leaving arc: Pick a tree arc having a negative (i.e. infeasible) primal flow.

Leaving arc: (g,a)
Entering arc: (d,e)

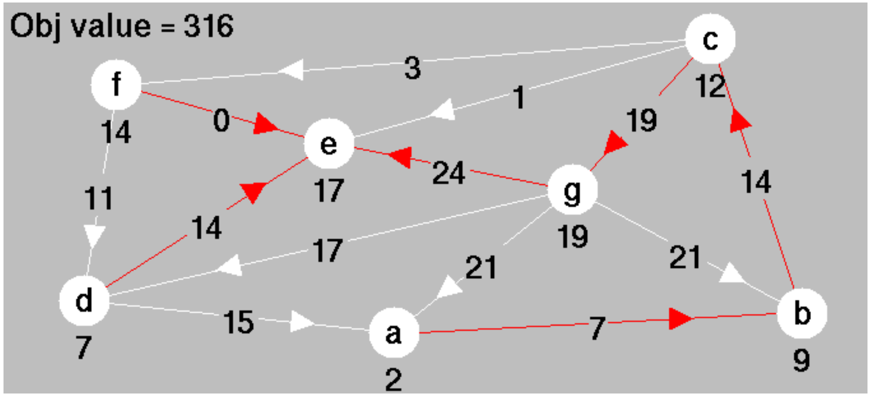


Entering arc: Remove leaving arc to split the spanning tree into two subtrees. Entering arc is an arc reconnecting the spanning tree with an arc in the *opposite* direction, and, of all such arcs, is the one with the *smallest* dual slack.

Dual Network Simplex Method—Second Pivot



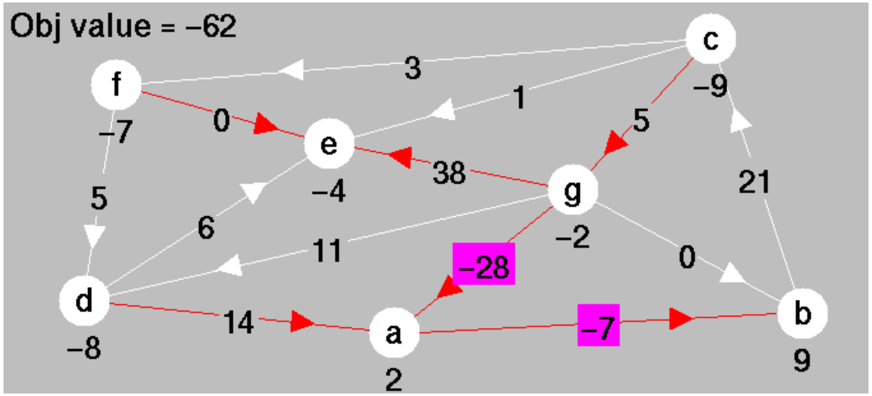
Leaving arc: (d,a)
Entering arc: (b,c)



Optimal!

Explanation of Entering Arc Rule

Recall initial tree solution:



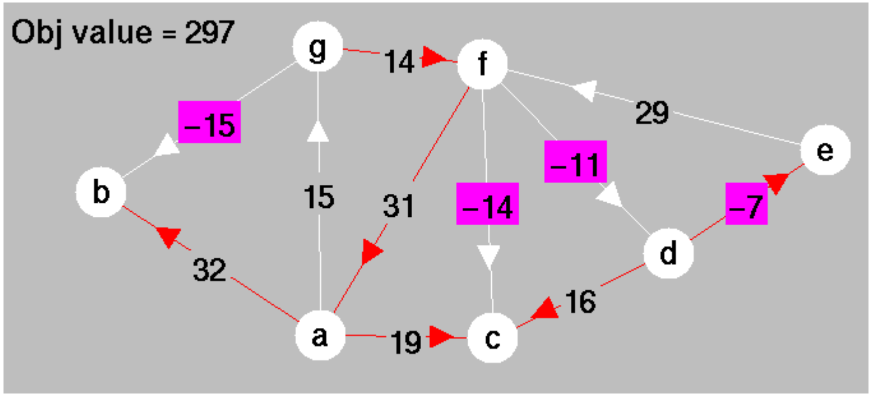
Leaving arc: (g,a)
Entering arc: (d,e)

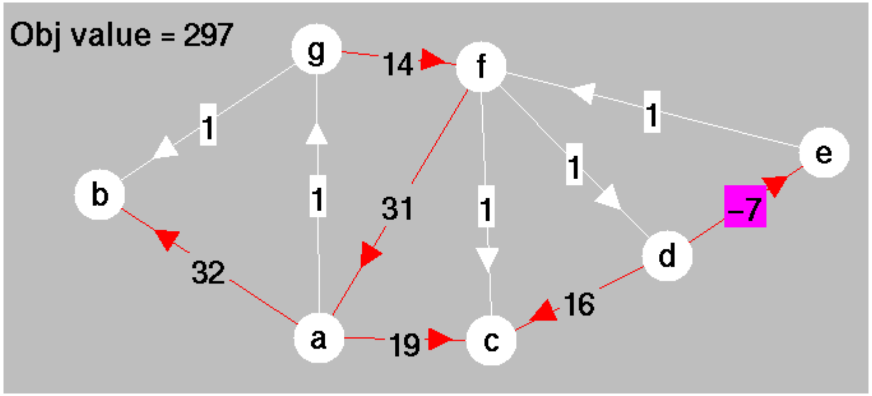
- Remove leaving arc. Need to find a reconnecting arc.
- Consider some reconnecting arc. Add flow to it.
  - If it reconnects in the same direction as leaving arc, such as (f,d), then flow on leaving arc decreases.
  - Therefore, leaving arc's flow can't be *raised* to zero.
  - Therefore, leaving arc can't leave. No good.

- Consider a potential arc reconnecting in the opposite direction, say (b,c).
  - Its dual slack will drop to zero.
  - All other reconnecting arcs pointing in the same direction will drop by the same amount.
  - To maintain nonnegativity of all the others, must pick the one that drops the least.

Two-Phase Network Simplex Method

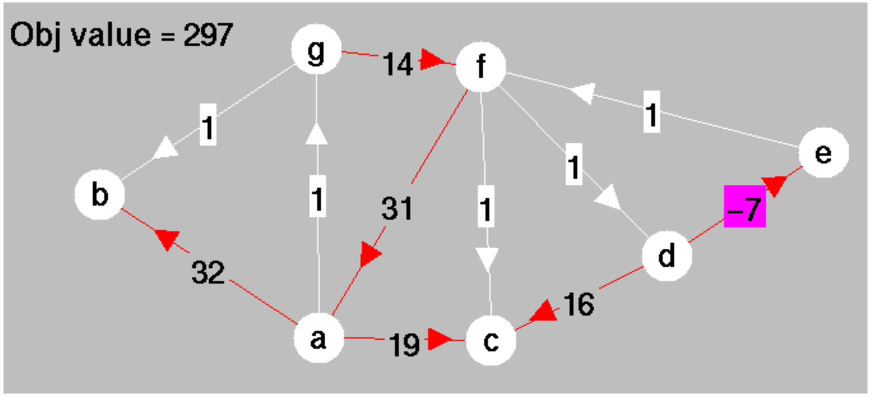
Example.



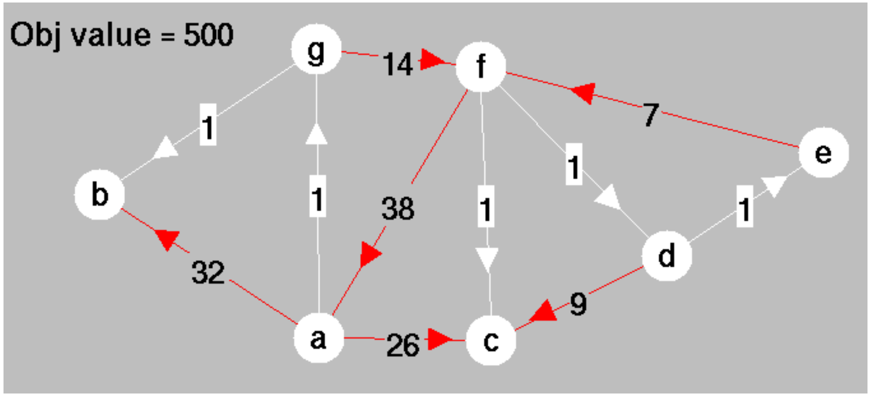


- Turn off display of dual slacks.
- Turn on display of artificial dual slacks.

Two-Phase Method–First Pivot

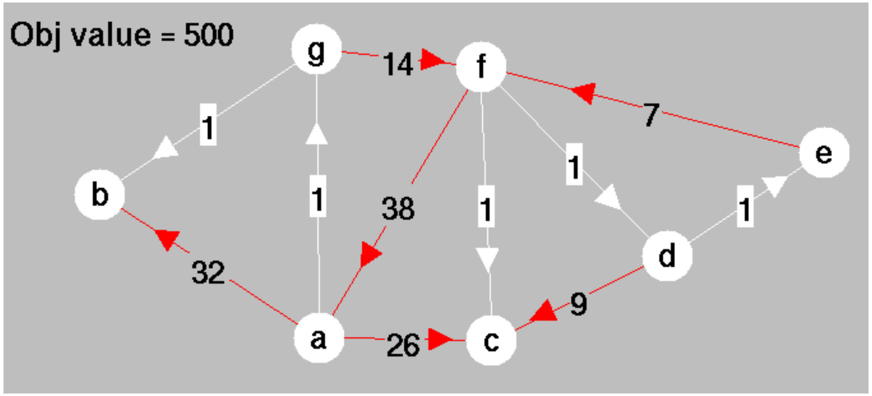


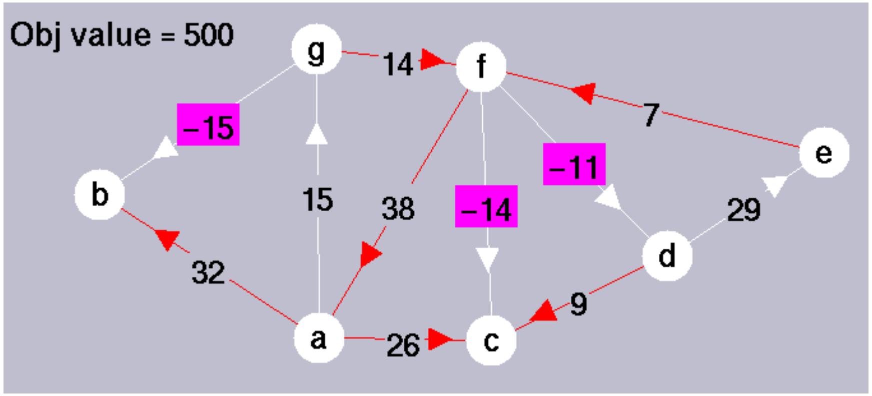
Use dual network simplex method.
Leaving arc: (d,e) Entering arc: (e,f)



Dual Feasible!

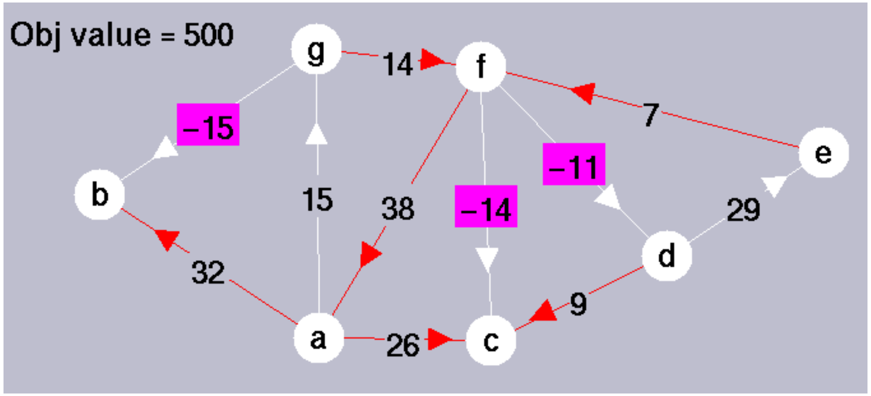
Two-Phase Method–Phase II



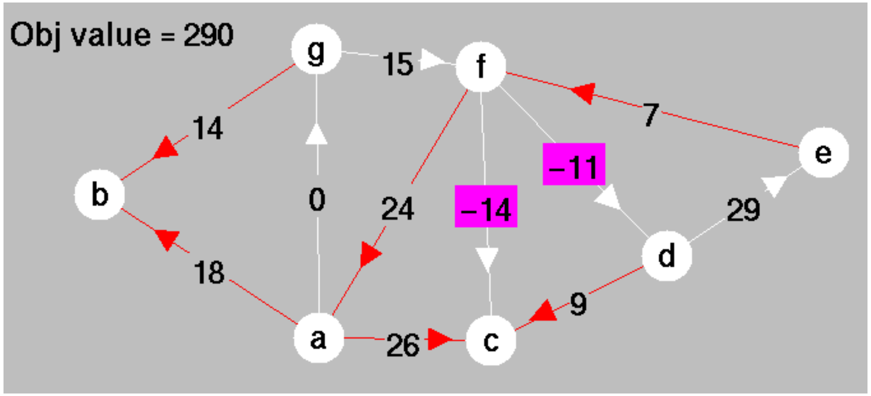


- Turn off display of artificial dual slacks.
- Turn on display of dual slacks.

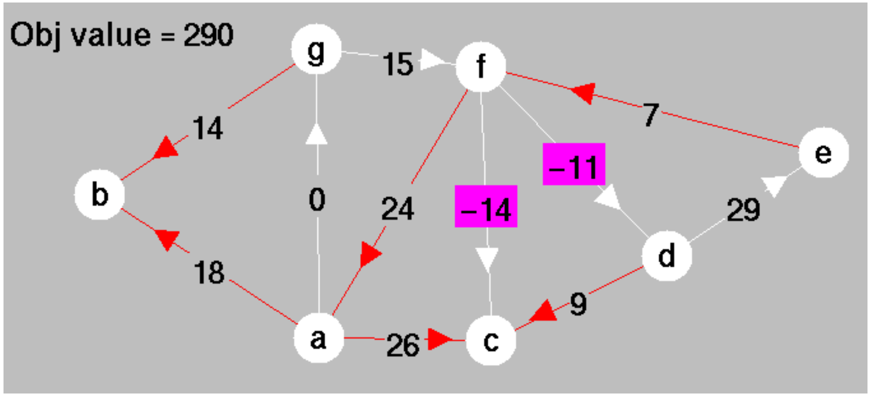
Two-Phase Method–Second Pivot



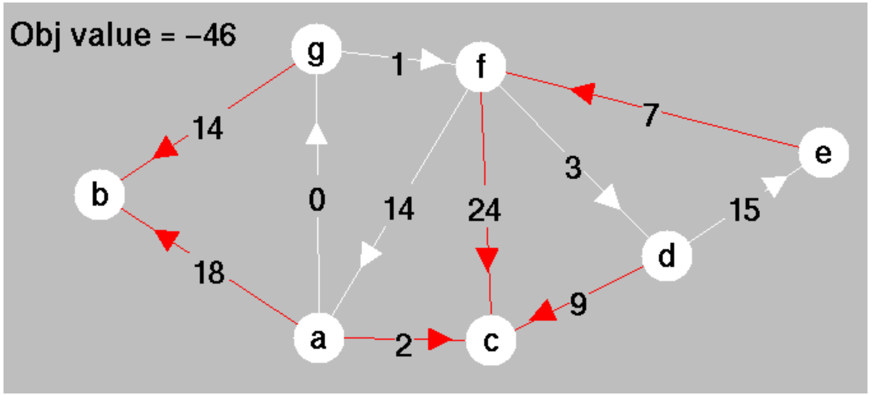
Entering arc: (g,b)
Leaving arc: (g,f)



Two-Phase Method–Third Pivot



Entering arc: (f,c)
Leaving arc: (a,f)



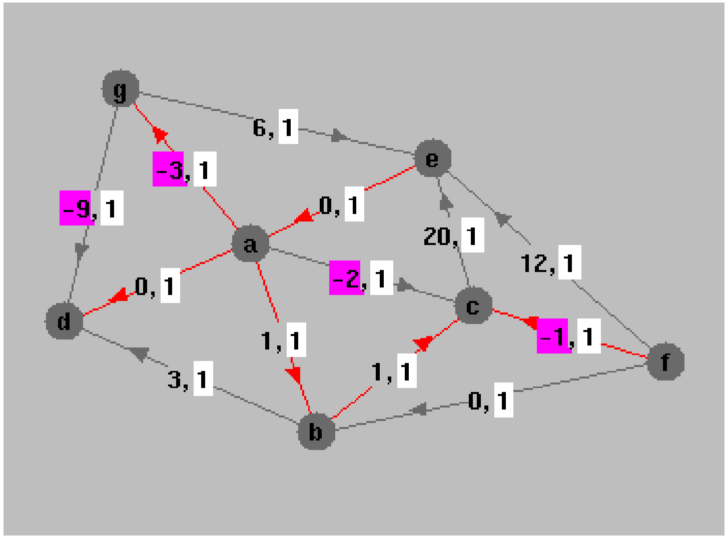
Optimal!

Online Network Simplex Pivot Tool

Click here (or on any displayed network) to try out the online network simplex pivot tool.

One-Phase Primal-Dual Method

- Artificial flows and slacks are multiplied by a parameter μ.
- In the Figure, $6, 1$ represents $6 + 1\mu$.
- *Question:* For which μ values is dictionary optimal?
- *Answer:*

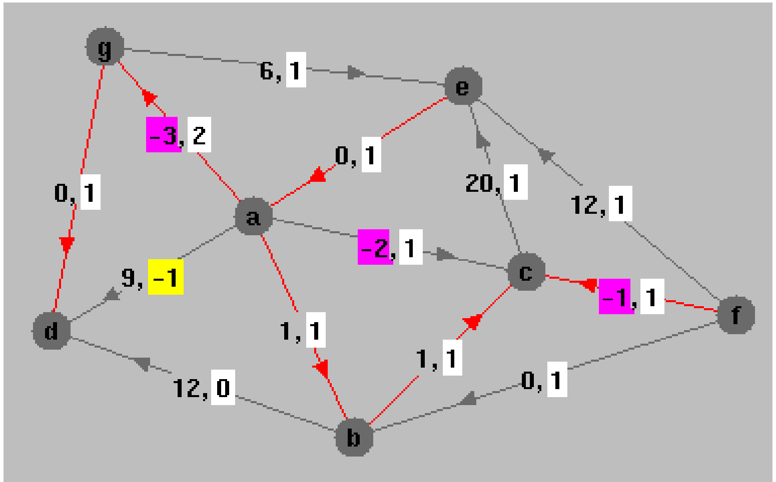


$$
\begin{array}{rcccll}
1 & + & \mu & \geq & 0 & (a,b) \\
-2 & + & \mu & \geq & 0 & (a,c) \\
 & & \mu & \geq & 0 & (a,d) \\
 & & \mu & \geq & 0 & (e,a) \\
-3 & + & \mu & \geq & 0 & (a,g) \\
 & & \mu & \geq & 0 & (b,c) \\
3 & + & \mu & \geq & 0 & (b,d)
\end{array}
\qquad
\begin{array}{rcccll}
 & & \mu & \geq & 0 & (f,b) \\
20 & + & \mu & \geq & 0 & (c,e) \\
-1 & + & \mu & \geq & 0 & (f,c) \\
-9 & + & \mu & \geq & 0 & (g,d) \\
12 & + & \mu & \geq & 0 & (f,e) \\
6 & + & \mu & \geq & 0 & (g,e)
\end{array}
$$

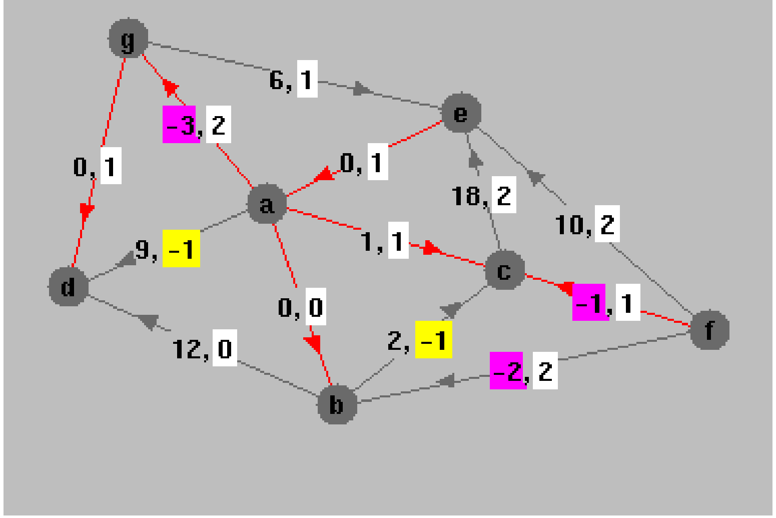
- That is, $9 \leq \mu < \infty$.
- Lower bound on μ is generated by arc (g,d).
- Therefore, (g,d) enters.
- Arc (a,d) leaves.

Second Iteration

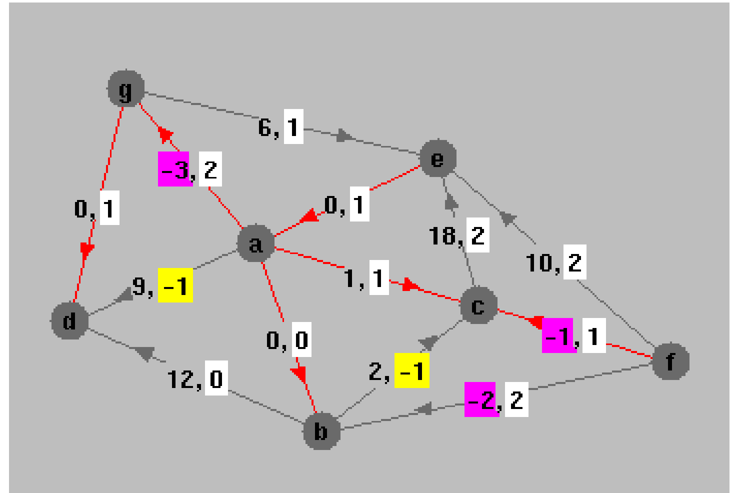
- Range of μ values: $2 \leq \mu \leq 9$.
- Entering arc: (a,c)
- Leaving arc: (b,c)



New tree:

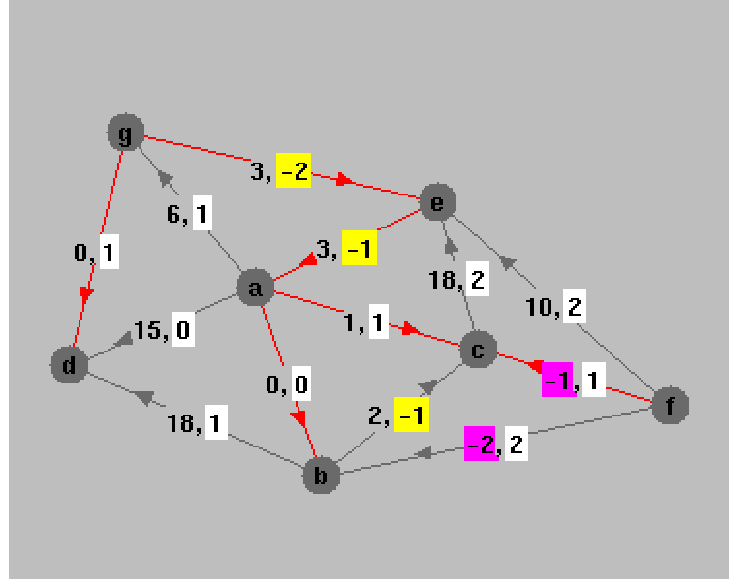


Third Iteration

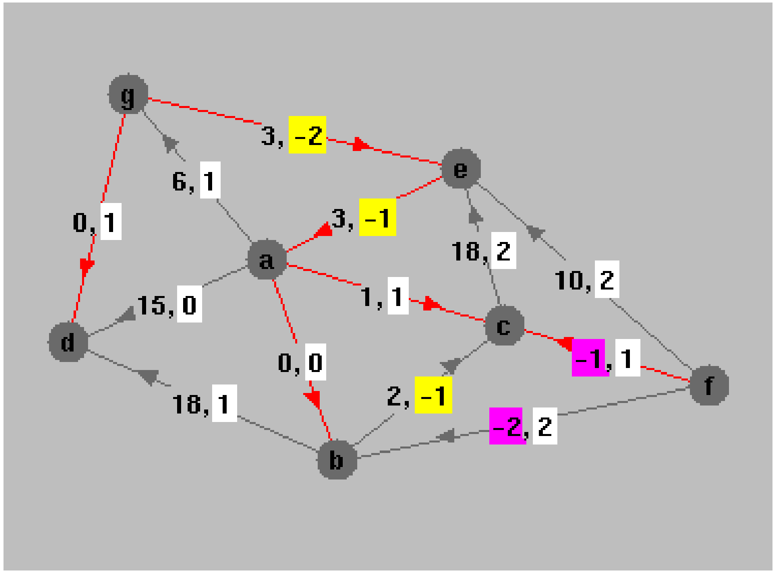


- Range of μ values: $1.5 \leq \mu \leq 2$.
- Leaving arc: (a,g)
- Entering arc: (g,e)

New tree:



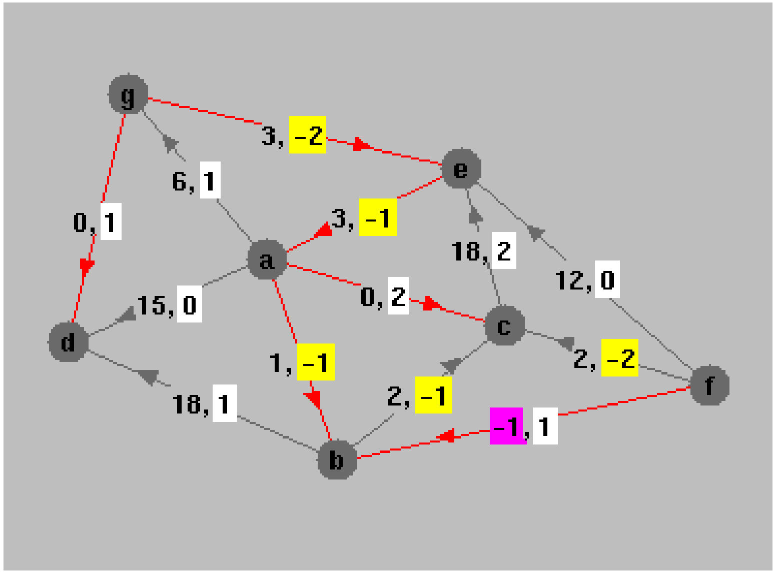
Fourth Iteration



- Range of μ values: $1 \leq \mu \leq 1.5$.
- A tie:
  - Arc (f,b) enters, or
  - Arc (f,c) leaves.
- Decide arbitrarily:
  - Leaving arc: (f,c)
  - Entering arc: (f,b)

Fifth Iteration

- Range of μ values: $1 \le \mu \le 1$.
- Leaving arc: (f,b)
- Nothing to Enter.



Primal Infeasible!

Online Network Simplex Pivot Tool

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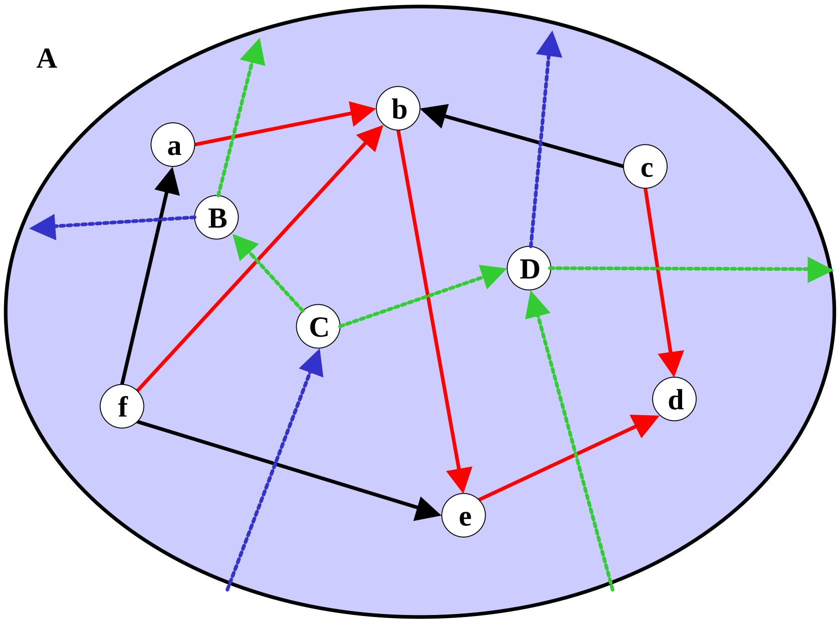
Planar Networks

Definition. *Network is called* planar *if can be drawn on a plane without intersecting arcs.*

Theorem. *Every planar network has a dual—dual nodes are* faces *of primal network.*



Notes:

- Dual node A is "node at infinity".
- Primal spanning tree shown in red.
- Dual spanning tree shown in blue (don't forget node A).

Theorem. *A dual pivot on the primal network is exactly a primal pivot on the dual network.*

Integrality Theorem

Theorem. *Assuming integer data, every basic feasible solution assigns integer flow to every arc.*

Corollary. *Assuming integer data, every basic optimal solution assigns integer flow to every arc.*