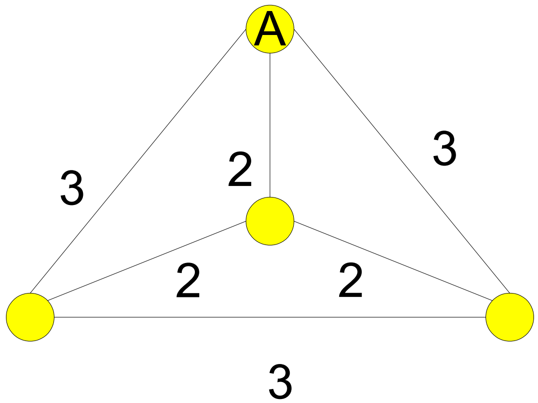
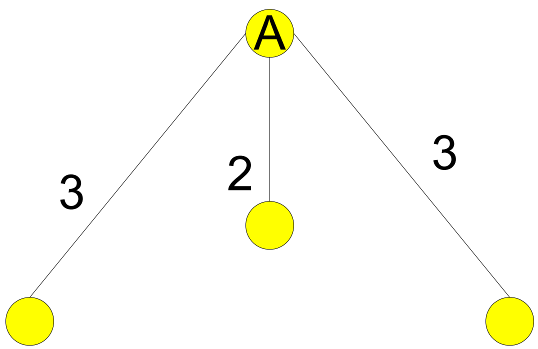
Minimum Spanning Tree

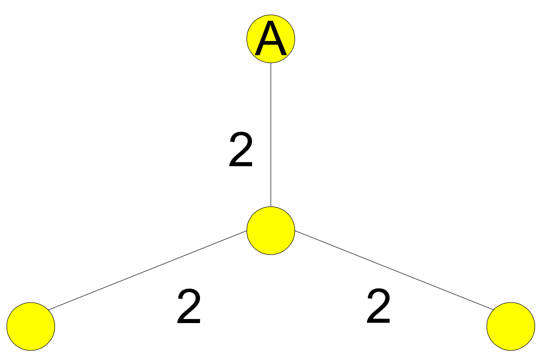
- Given a weighted graph G, we want to find the least-cost tree that spans the graph.

MST vs SPT





Shortest path tree from A
Total Cost: 8
Total Cost of Paths from A:
3+3+2=8



Minimum Spanning tree
Total Cost: 6
Total of Paths from A:
2+4+4=10

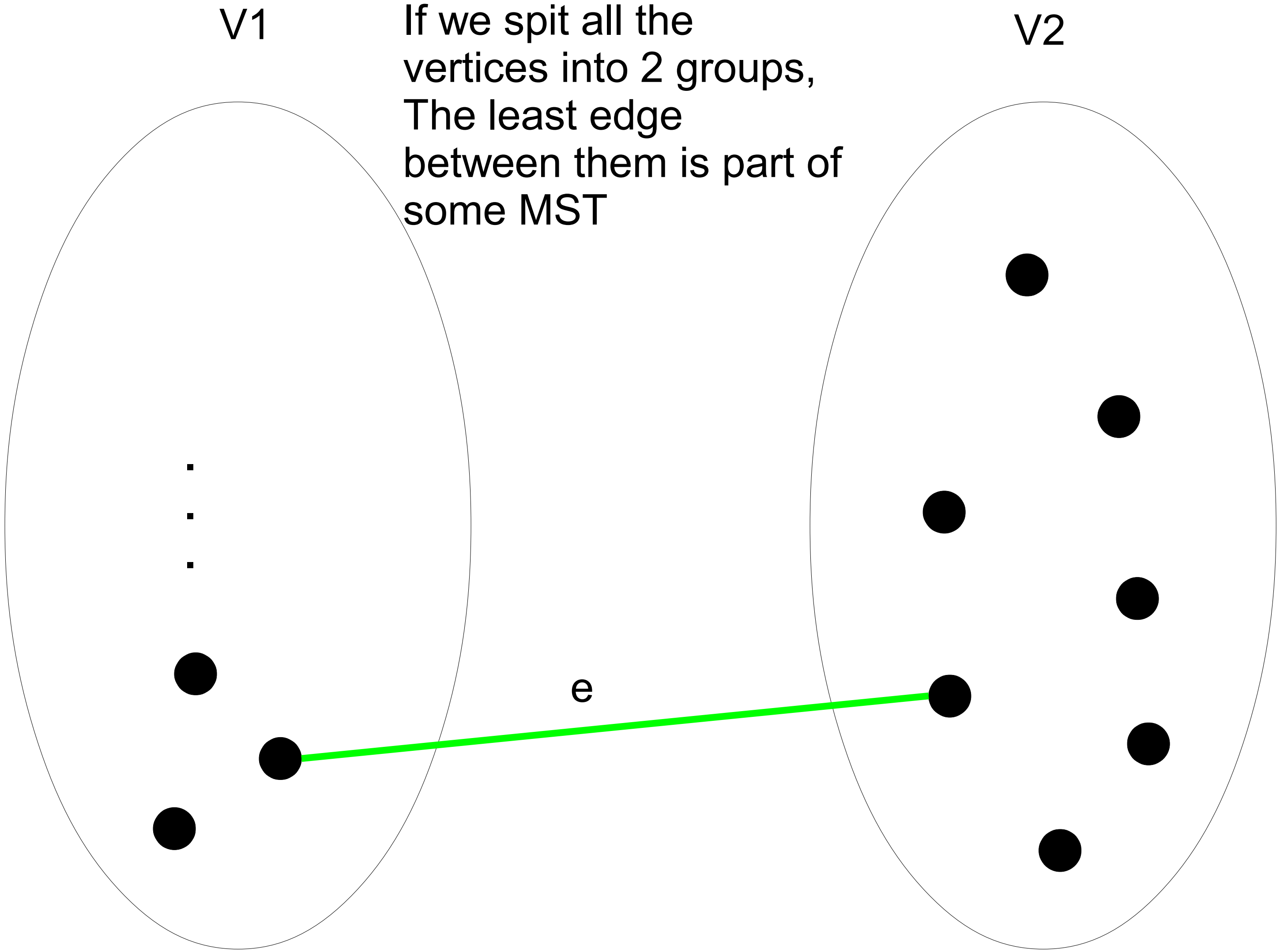
Kruskal's Algorithm

- One way to find a MST is via Kruskal's algorithm:
- Take the smallest edge that does not induce a cycle, and insert it into our subgraph.
- Do this until all nodes are connected

Kruskal's Algorithm

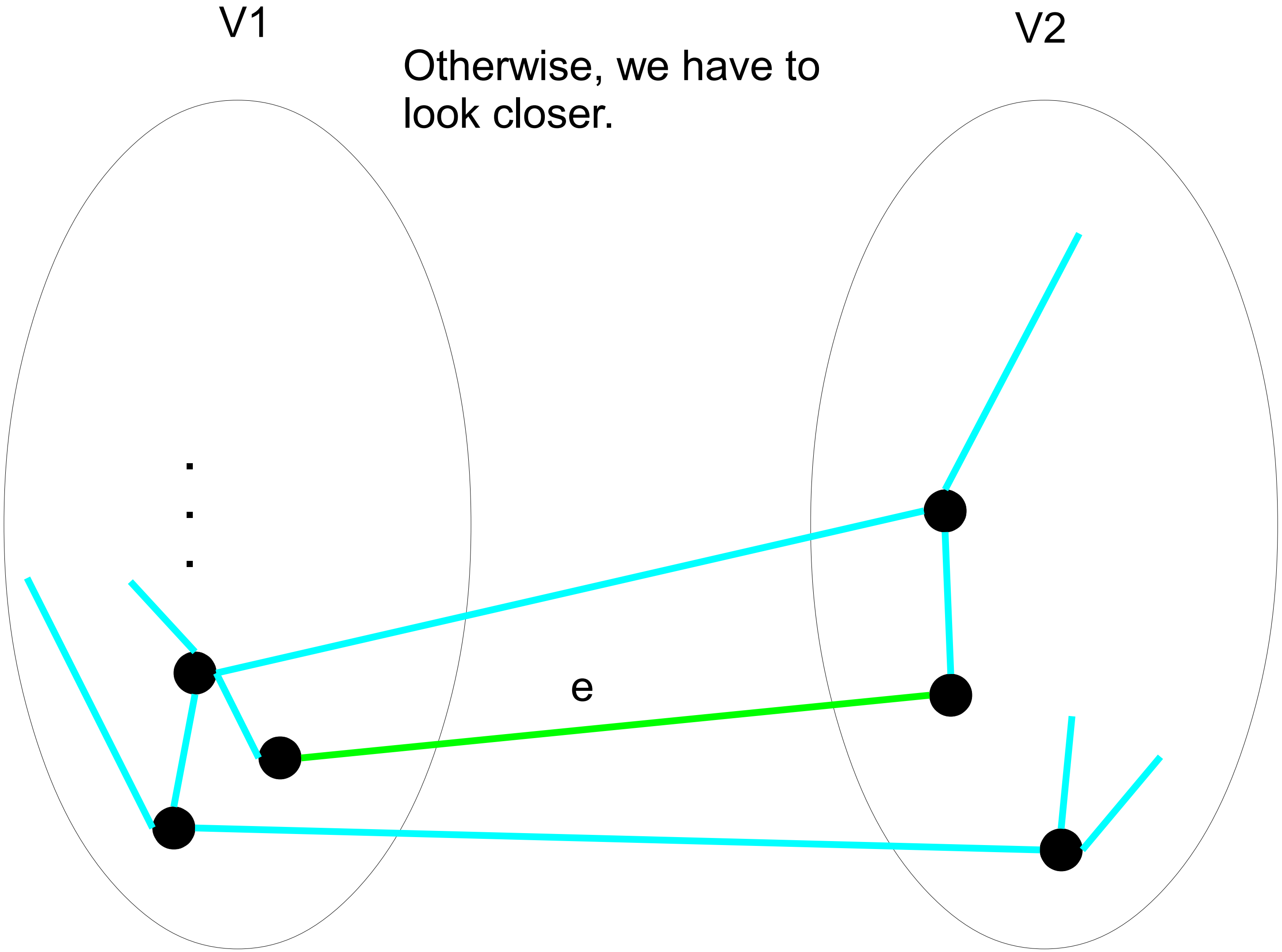
- One way to find a MST is via Kruskal's algorithm:
- Take the smallest edge that does not induce a cycle, and insert it into our subgraph.
- Do this until all nodes are connected
- A naive way to make sure an edge does not induce a cycle is by using DFS or BFS from one of the edge's vertices, and seeing if we reach the other. If we do, adding that edge would create a cycle.

Remember the Cut Property:
If we spit all the vertices into 2 groups, The least edge between them is part of some MST



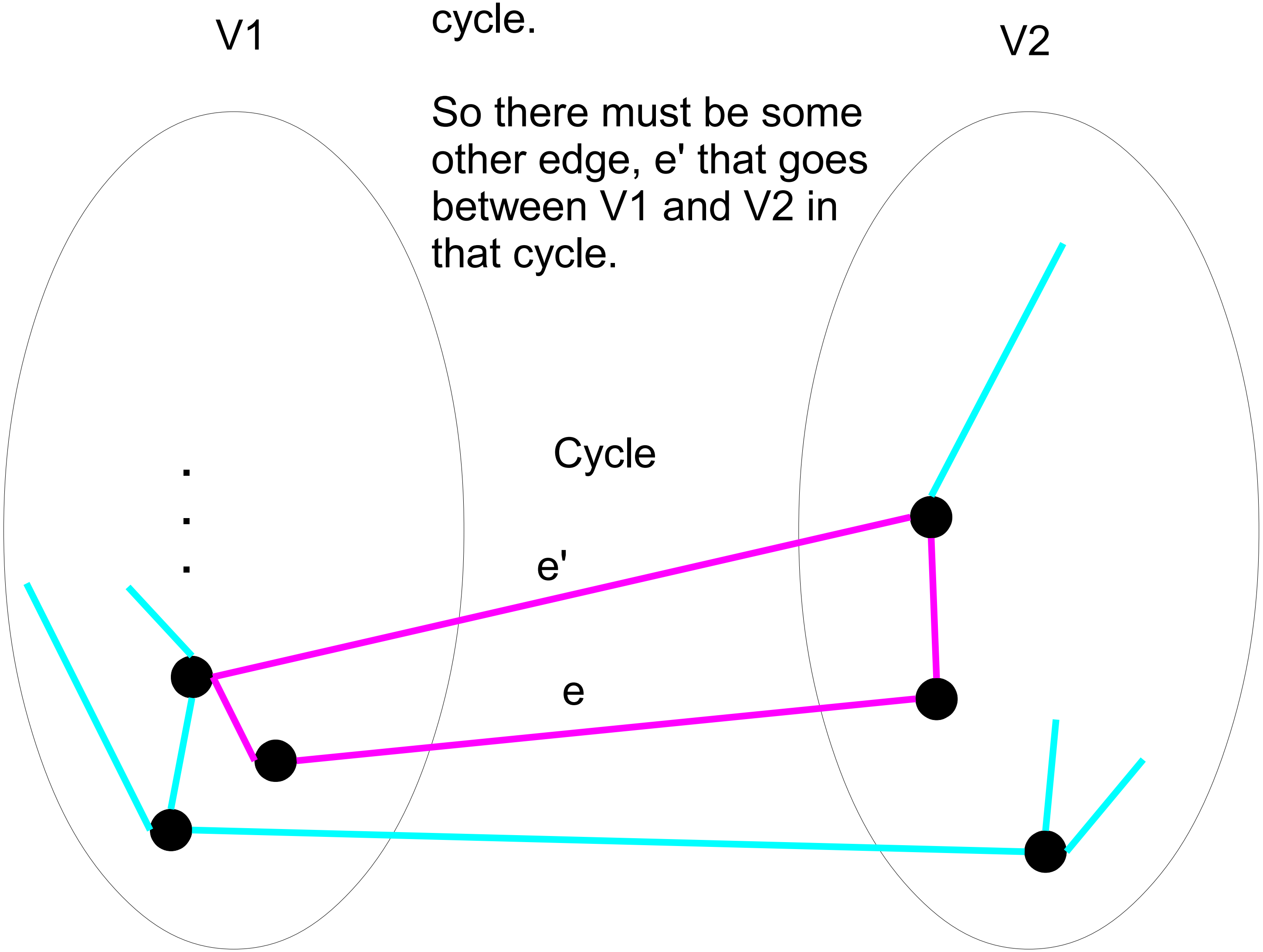
If we have a MST with e, we are done.

Otherwise, we have to look closer.



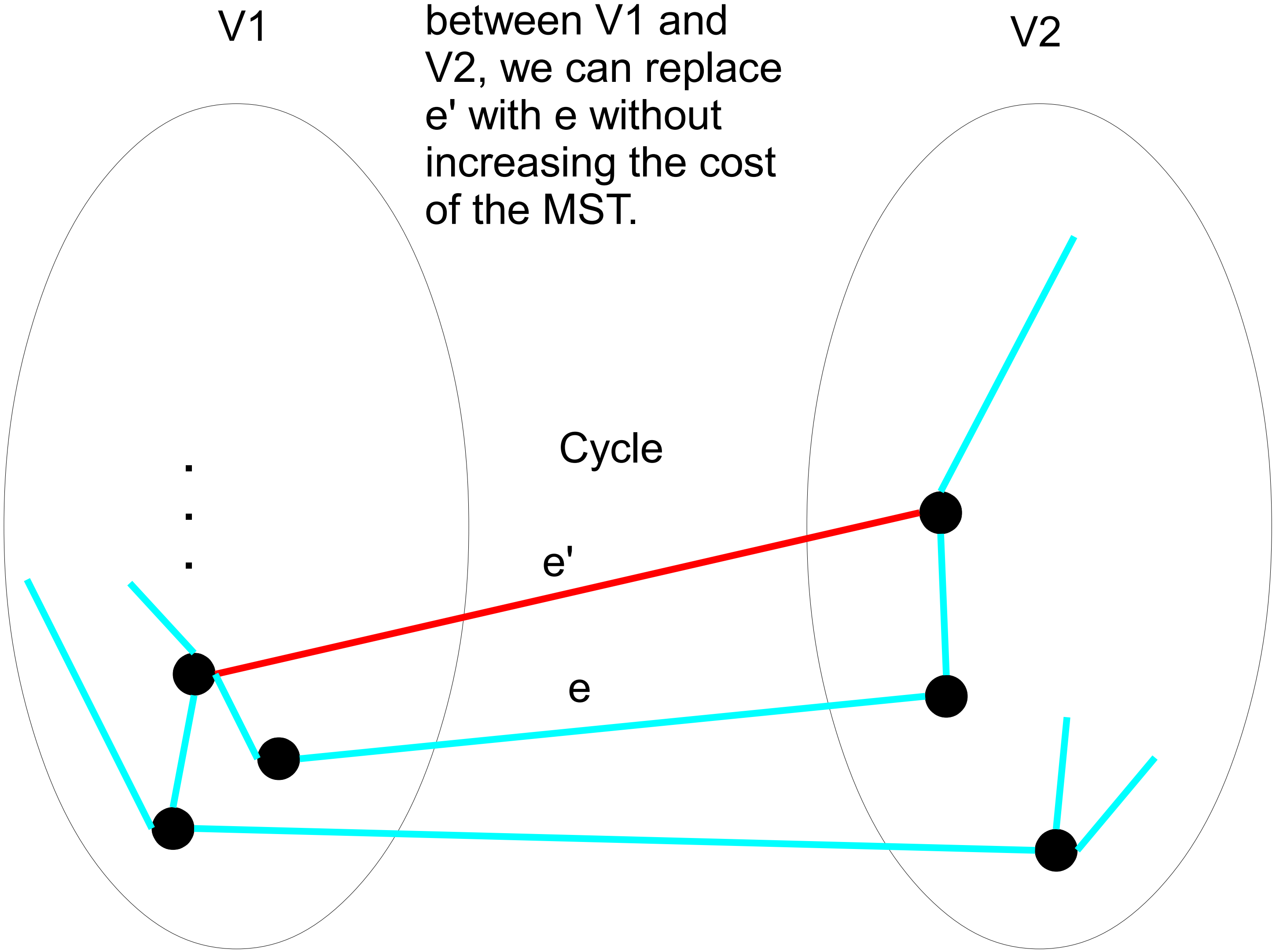
Since we had a MST, adding e creates a cycle.

So there must be some other edge, e' that goes between V1 and V2 in that cycle.

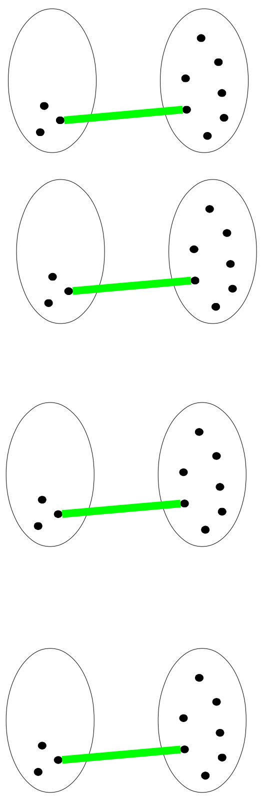


(Since it's a cycle, we have to go from V1 to V2, and then back again)

However, since e is
the least edge
between V1 and
V2, we can replace
e' with e without
increasing the cost
of the MST.

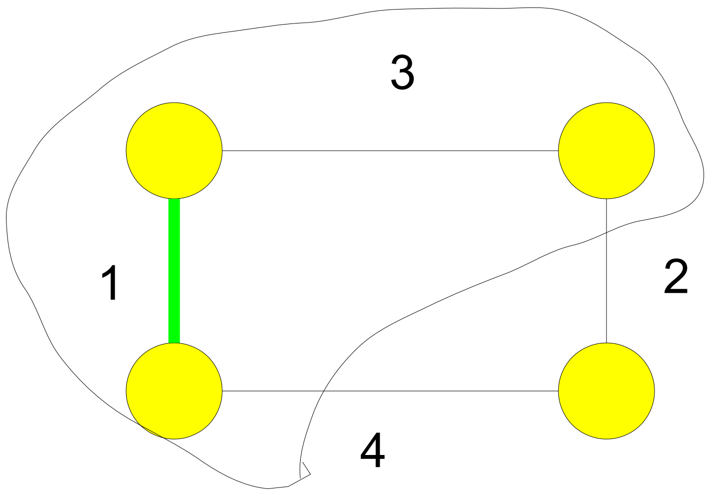


The graph is also connected because removing one edge from a cycle never disconnects the graph.



In Kruskal's algorithm, we adding the least edge e that does not form a cycle.

In other words, e is the least edge of some cut where V1 is a connected component, and V2 is the rest of the edges.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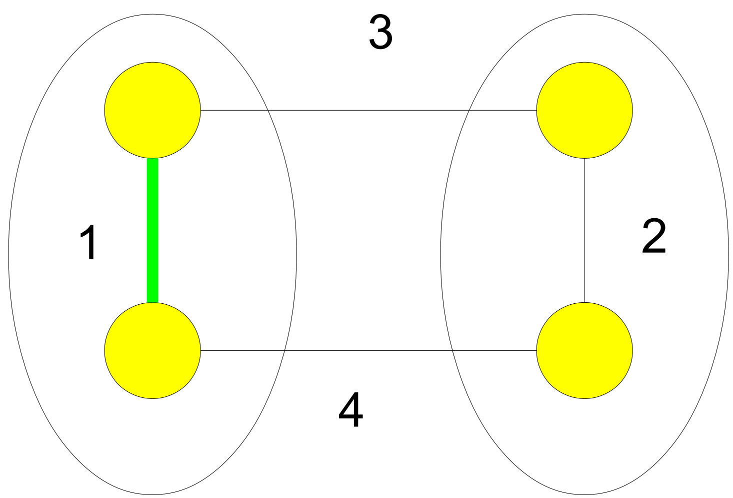
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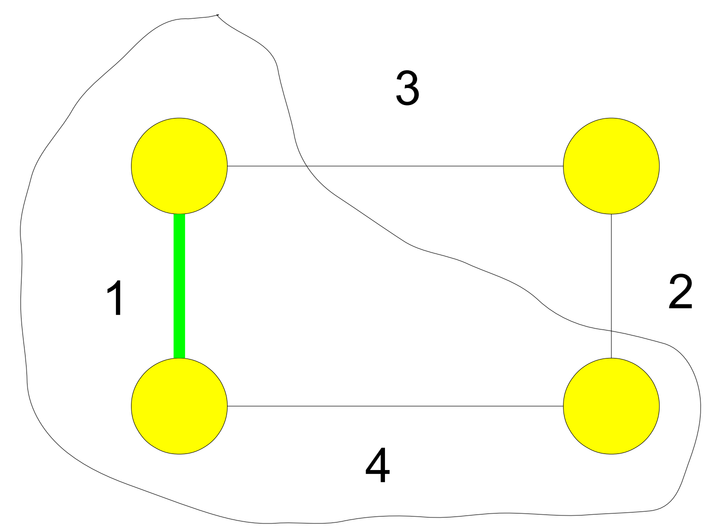
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Priority Queue

- Then, we can just use a Priority Queue to store the edges, since we only want the current cheapest one.
- However, we may poll an edge that is cheapest, but forms a cycle

Cycles

- The least cost edge is an edge between two connected components.
- So we want to ignore and edge if it is incident to two vertices in the same component.

Connected Components

- So all we have to do is keep track of the connected components we have formed.
- The best way to do this is with a Union-Find data structure
  - These are in your book

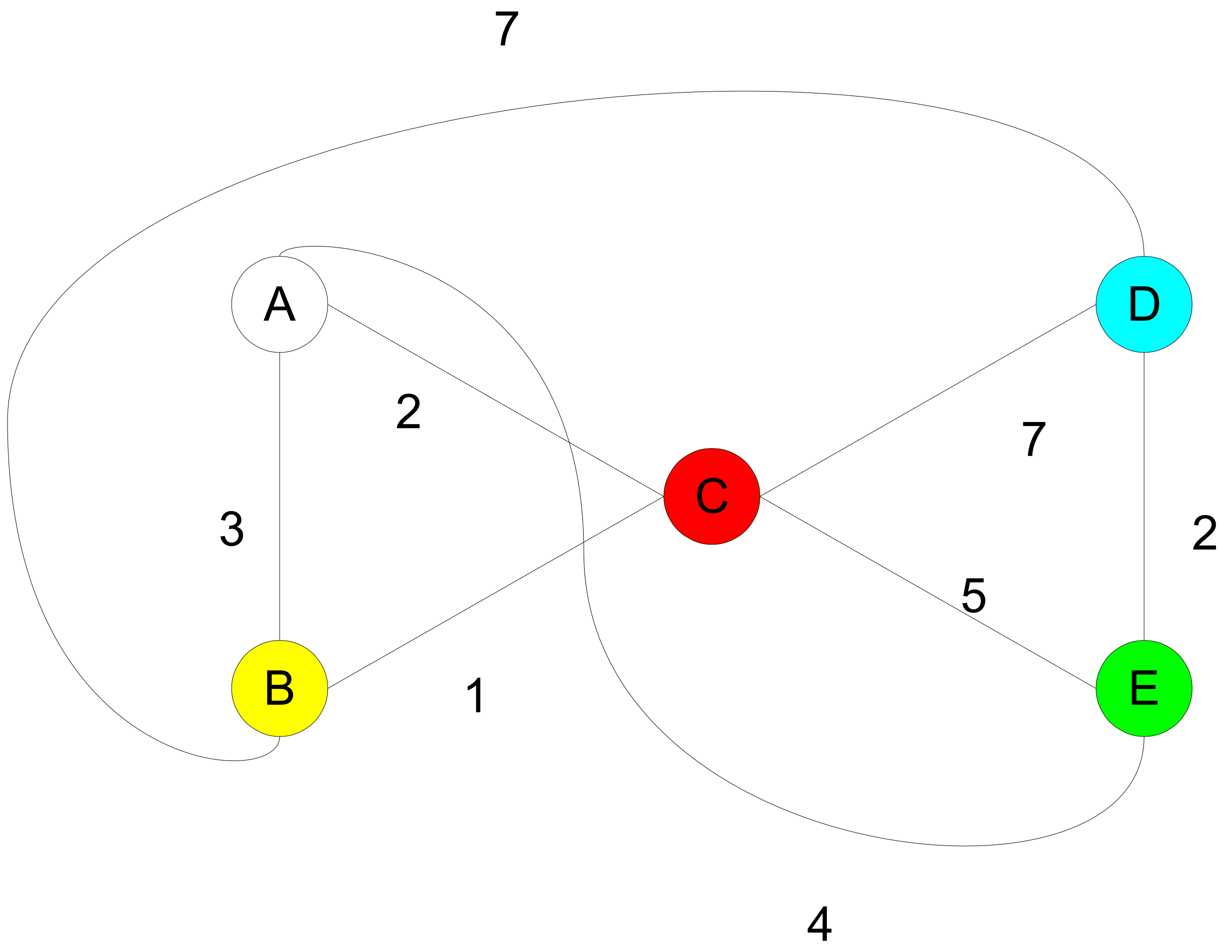
Simple Union-Join

- There are more efficient ways, but for our purposes we will use an array
- What we can do is have an array that has an entry for every vertex.
- The entry corresponds to which component the vertex belongs to

Simple Union-Find

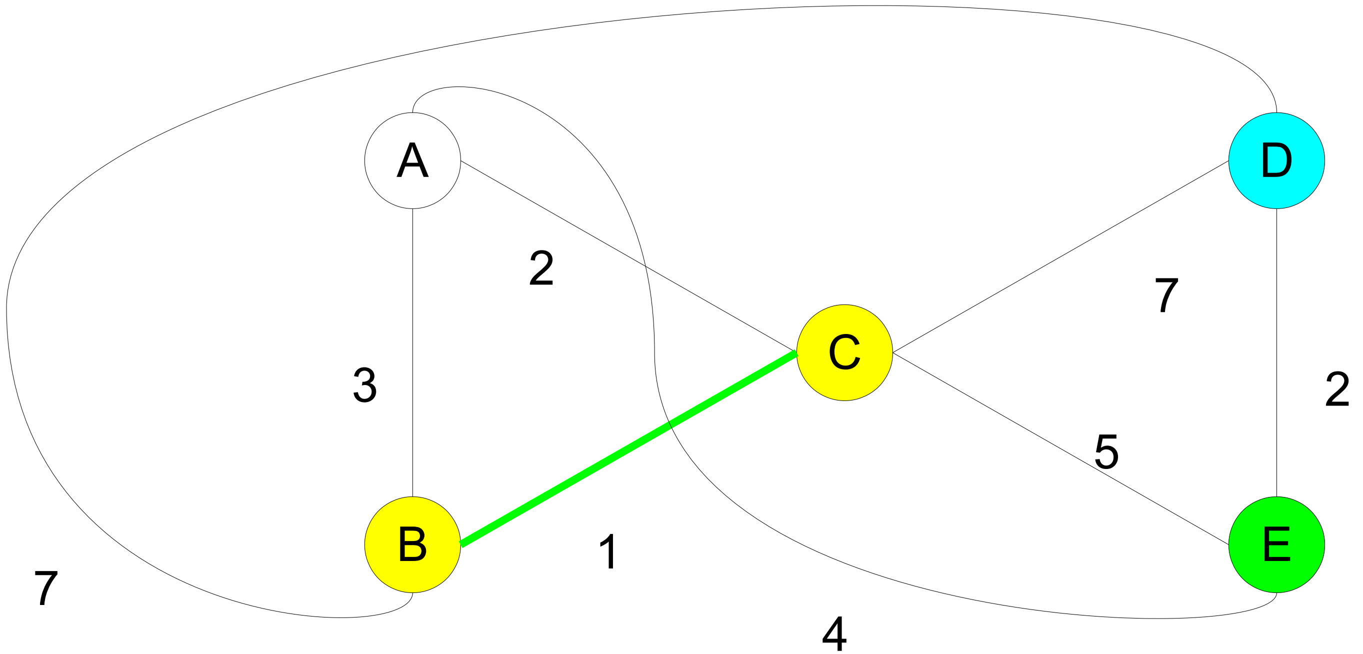
- Initially, each entry is just the index of the array (each vertex is its own component)
- When we connect two components together, with numbers x and y.
- We then iterate through the array, replacing each y with x.

```
init:
         for (each node k) do
               groupID[k] = k;         // groupID[k] = id of the group that node k
belongs
         Edges = queue of edges ordered by the cost of the edge

Kruskal's Algorithm:
         while ( not all nodes included ) {
            e = next edge in Edges;     (least cost unprocessed edge)
            if ( e connects 2 vertices of the same group)
                   discard edge;
              else               {   // e connect 2 different groups of nodes together
                   Add e to MST;
                 G1 = group ID of one of the groups connected by e;
                 G2 = group ID of the other group connected by e;

                for ( each node k with groupID == G2)
                         groupID[k] = G1;         // Put node in group G2 into group
G1
                }
         }
```



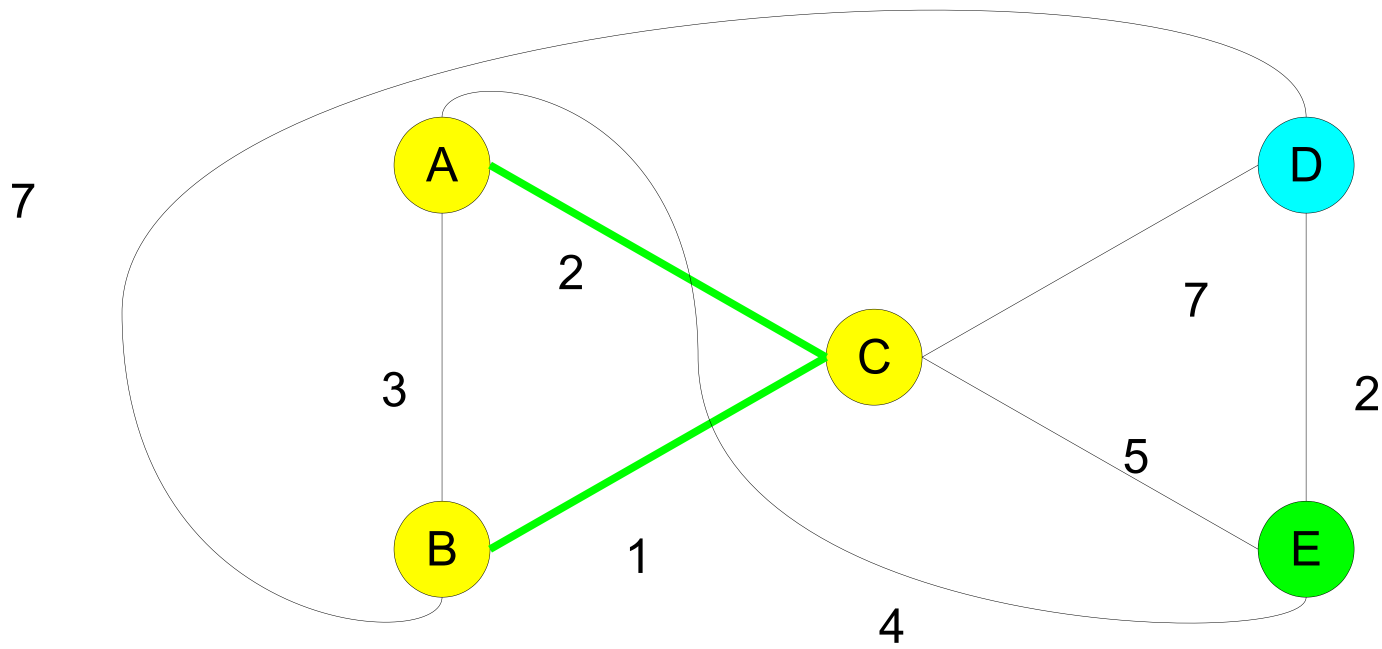
Edges: { (BC,1),(AC,2),(DE,2),(AB,3),(AE,4),(CE,5),(BD,7),(CD,7)}

Vertex Groups: {{A}, {B}, {C}, {D}, {E}}



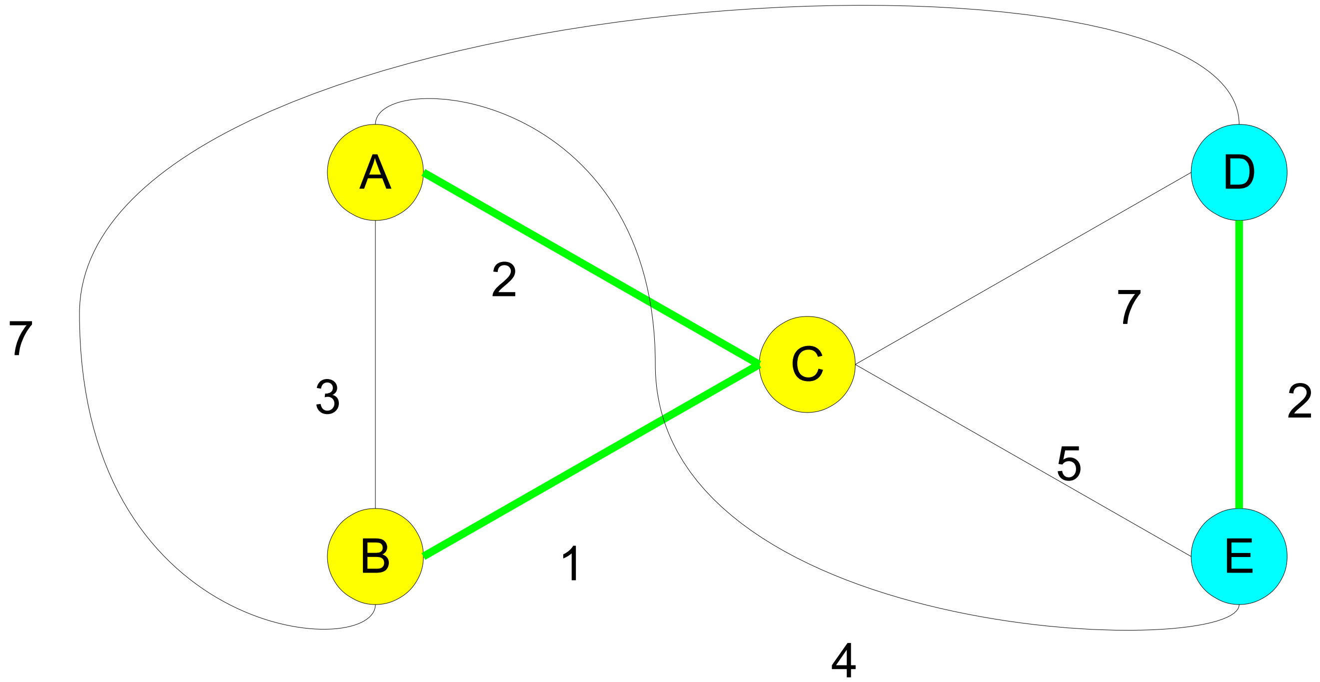
Edges: {(AC,2),(DE,2),(AB,3),(AE,4),(CE,5),(BD,7),(CD,7)}

Vertex Groups: {{A}, {B,C}, {D}, {E}}



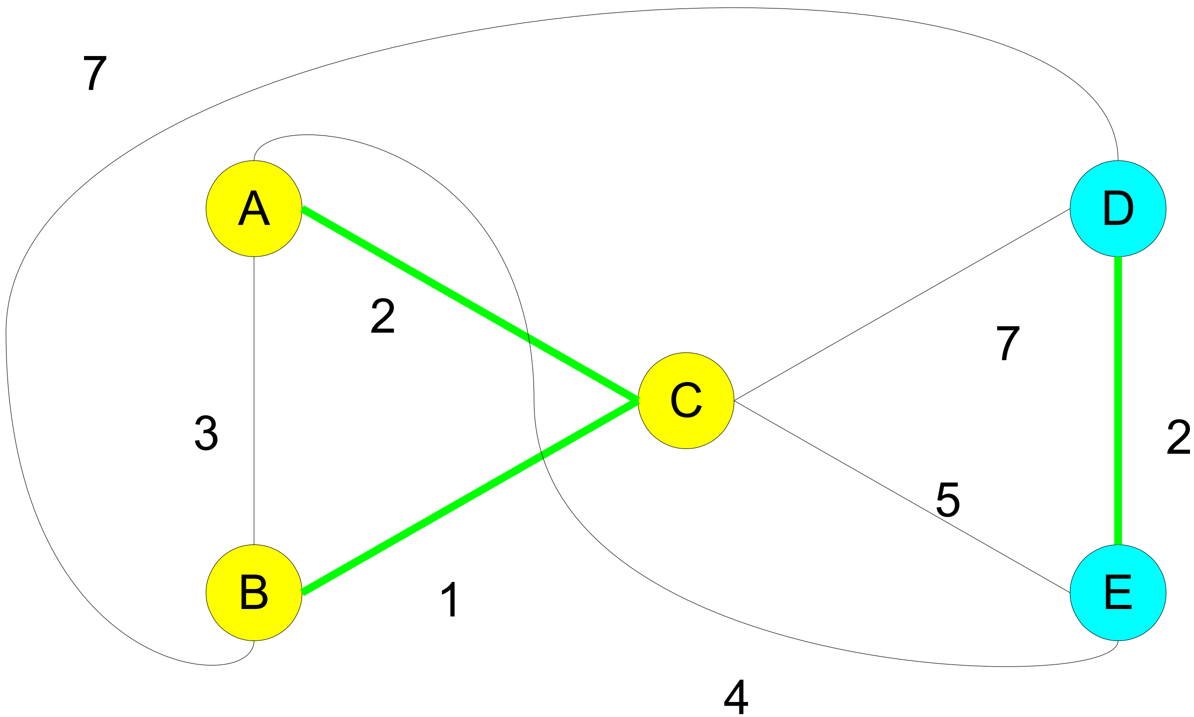
Edges: {(DE,2),(AB,3),(AE,4),(CE,5),(BD,7),(CD,7)}

Vertex Groups: {{A,B,C}, {D}, {E}}



Edges: {(AB,3),(AE,4),(CE,5),(BD,7),(CD,7)}

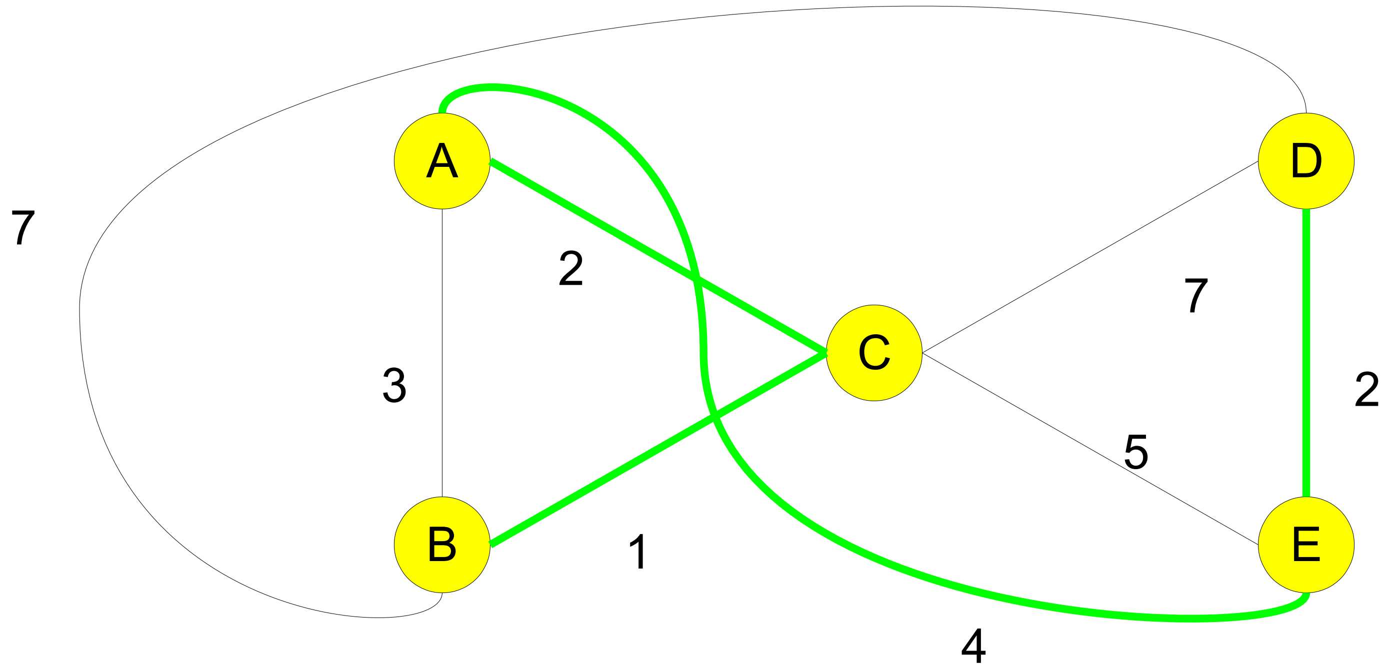
Vertex Groups: {{A,B,C}, {D,E}}



Edge AB ignored because A and B are part of the same connected component.

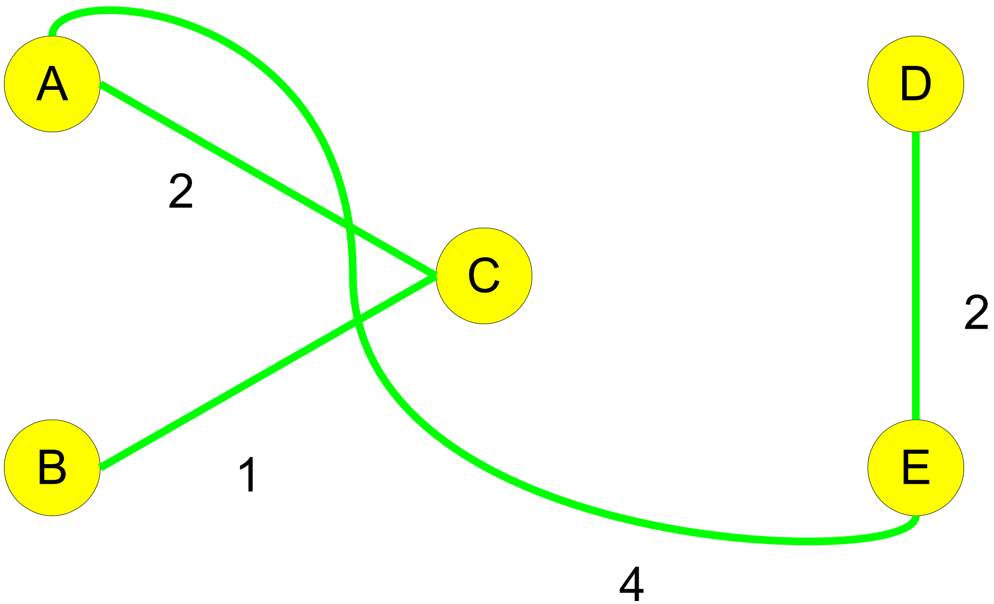
Edges: {(AE,4),(CE,5),(BD,7),(CD,7)}

Vertex Groups: {{A,B,C}, {D,E}}



Edges: {(CE,5),(CD,7)}

Vertex Groups: {{A,B,C,D,E}}



End tree

Run Time

- Run time of Kruskal's: n vertices, m edges. Assuming heap for priority queue
- Priority Queue operations O(mlog(m)) for insertions, but there is a linear way to do it as well.
- At worse, we need to remove all edges from the PQ, which is O(mlog(m))

Run Time

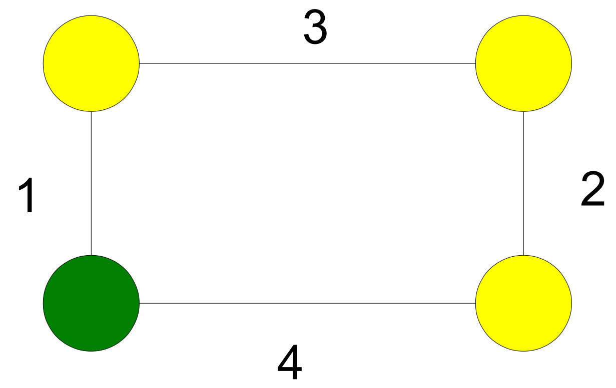
- Since the graph is simple, the number of edges is at most n^2/2 which we'll simplify to n^2.
- So our removal is O(mlog(n^2))=O(2mlog(n))=O(mlog(n))

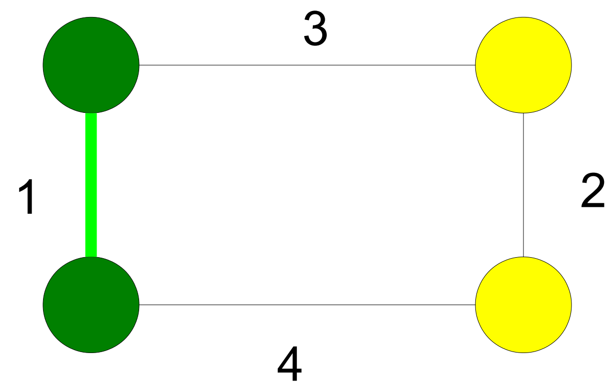
- Using a union-join data structure, we can form clusters and query clusters in mlog(n) time.
- So the total run time is O(mlog(n))

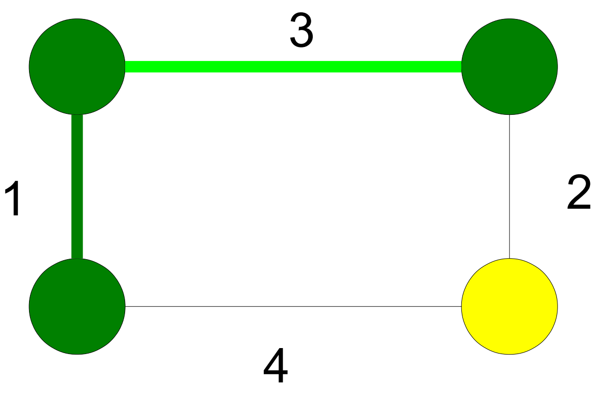
Prim's Algorithm

- Mark a vertex.
- while we still don't have a spanning tree
- Take the least edge that is between a marked and unmarked vertex
- mark the unmarked vertex

Simple Prim's



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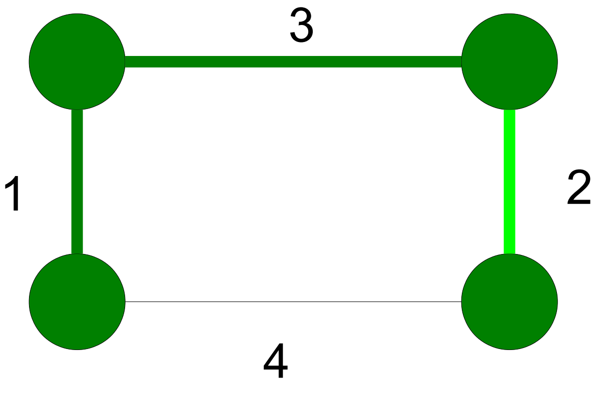
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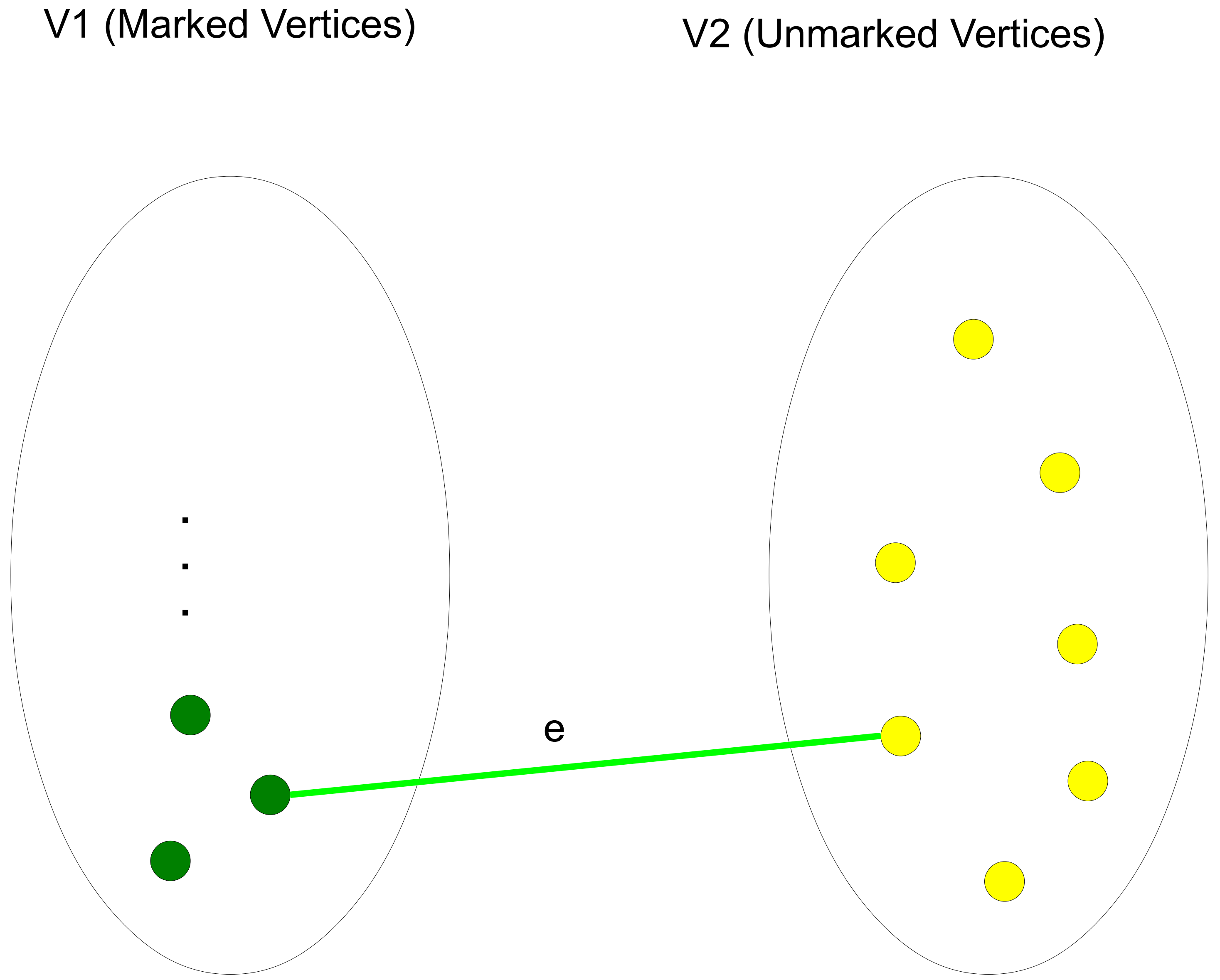

V1 (Marked Vertices)
V2 (Unmarked Vertices)
e

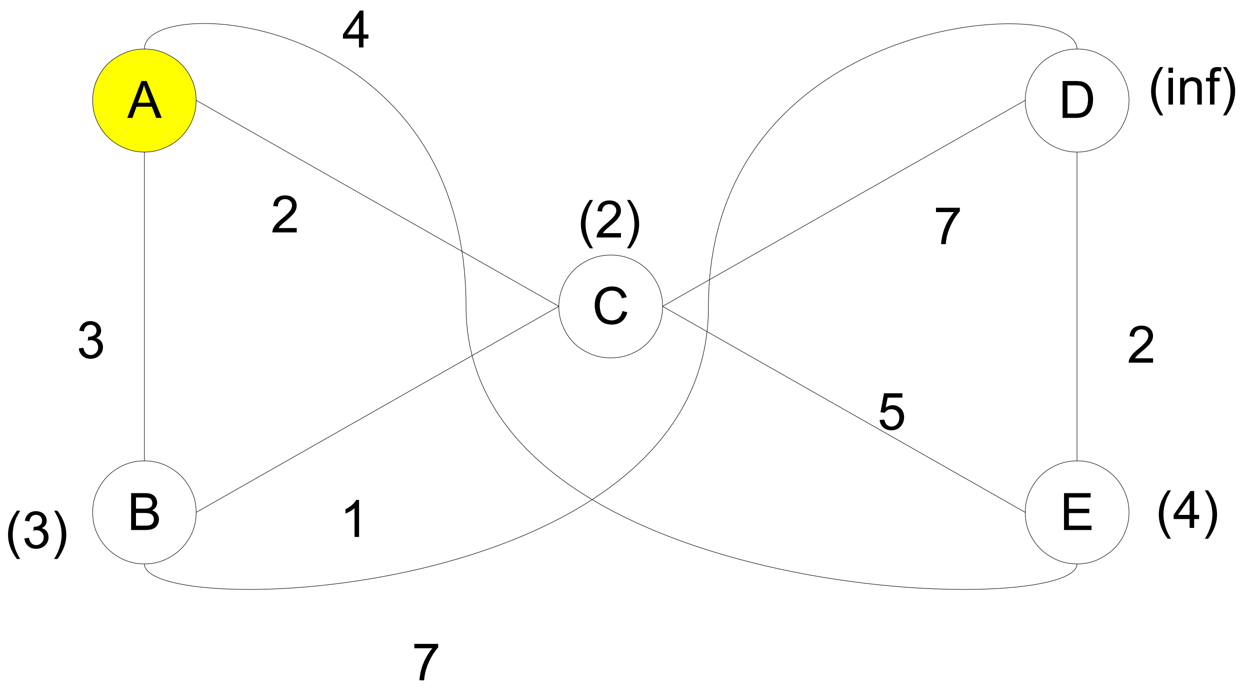
Implementation Notes

- To implement Prim's algorithm, we take some ideas from Dijkstra's
- We give each vertex a label corresponding to the weight of the least edge connected to a marked vertex.
- Since we always want the least, we can use a priority queue.
  - But as we mark vertices, the label can change.
  - So we want to use an adaptable priority queue.

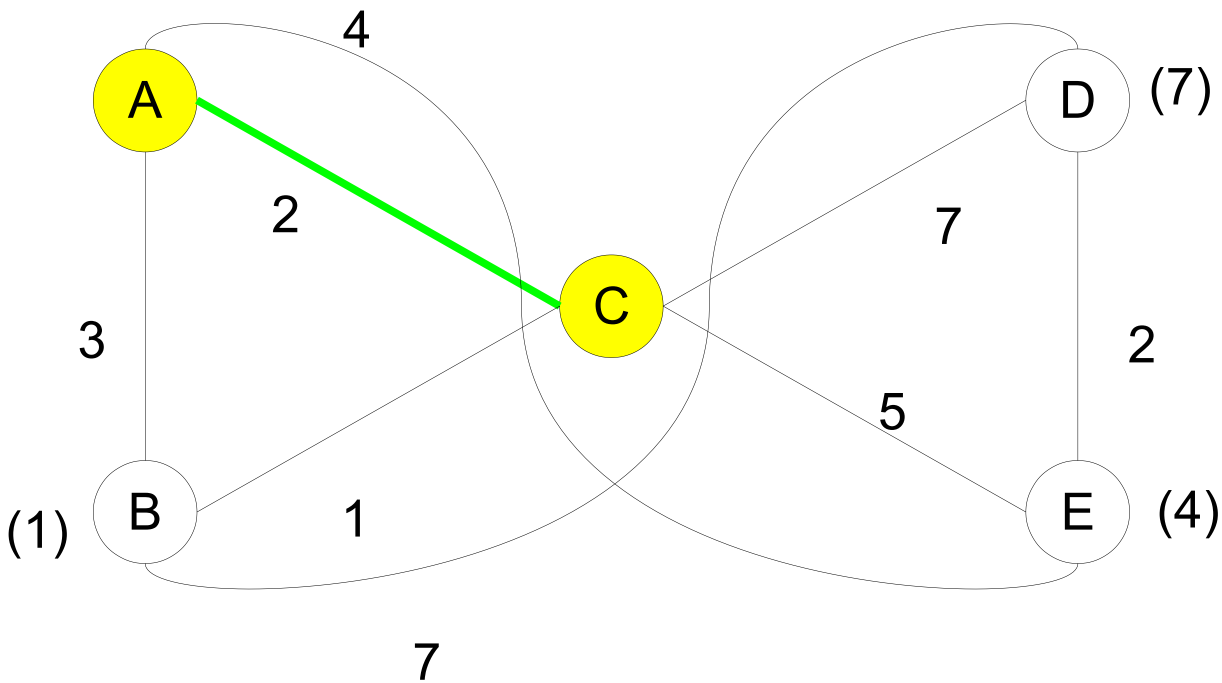
Vertex Label Updates

- When we mark a vertex, we iterate through all of its edges and update each unmarked vertex.

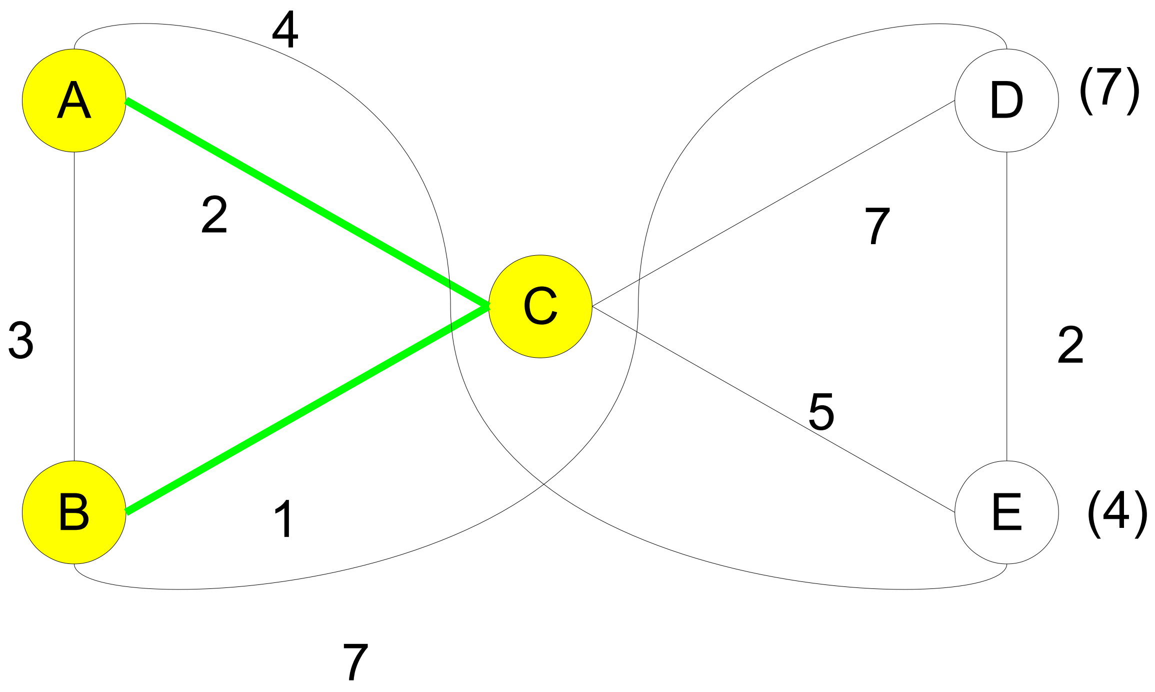
```
Init:
    For each vertex v:
        Label v infinity
    Vertices = all vertices of the graph ordered by the label

Prims Algorithm:
    while(Vertices is not empty):
        V = next vertex in Vertices //Least cost vertex

        Add V to the subgraph;

        if(V has an edge) :
            Add the edge to the subgraph; //The first added vertex
will not have a corresponding edge


        for(each edge e that contains V) :
            V2 = vertex in e that is not V;
            if(e.cost < label of V2) :
                V2's label = e.cost;
                V2's edge = e;
```



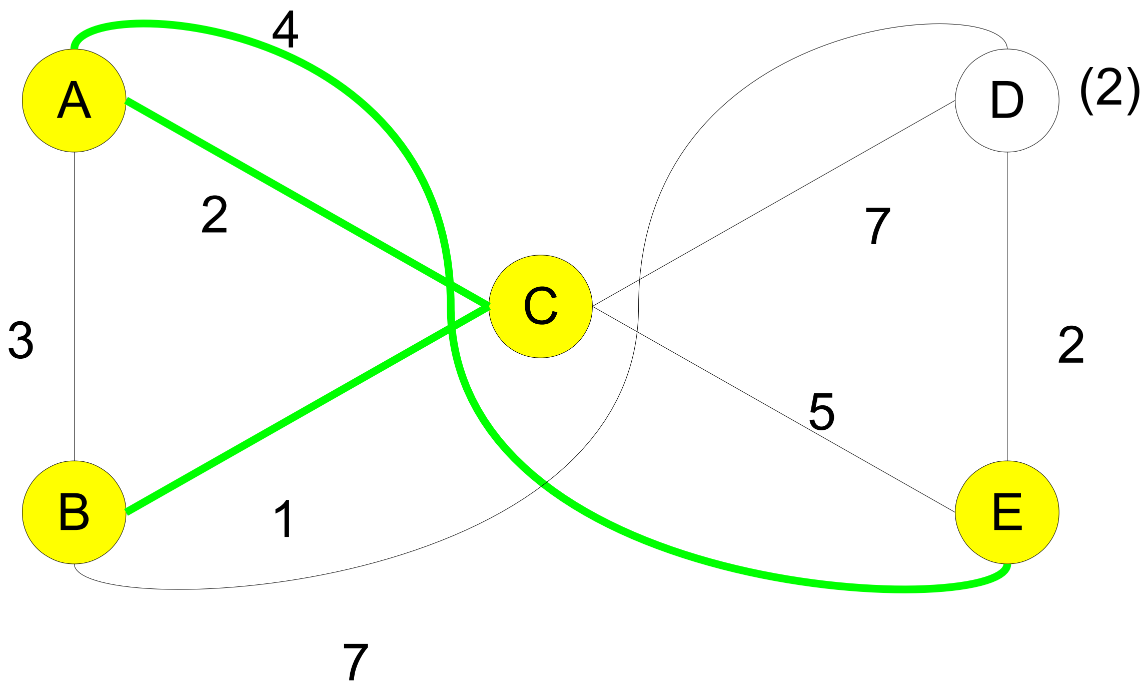
PQ: [(C,2), (B,3), (E,4), (D,infinity)]



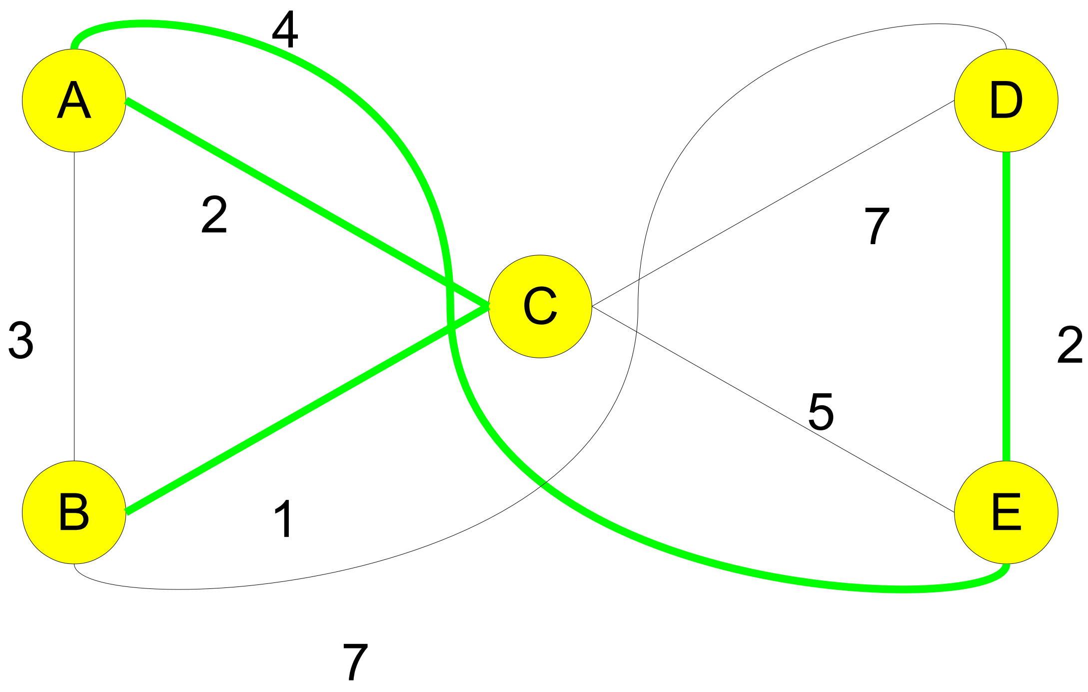
PQ: [(B,1), (E,4), (D,7)]



PQ: [(E,4), (D,7)]



PQ: [(D,2)]



A
B
C
D
E
4
2
3
1
7
7
5
2

Prim's Run Time

- We always need to remove all vertices from the PQ, which is O(nlog(n))
- We may have to update a vertex on every edge it has, which is O(mlog(n)).
  - M priority queue updates, each which is log(n)
- So the total running time is O((m+n)log(n)).
- Since m is either close to n (since the graph must be connected) or much greater than n (up to O(n^2)), we can write this as O(mlog(n))