Abstract: Let $E$ be an elliptic curve defined over $\mathbb{Q}$ without complex multiplication and let $\ell$ be a prime. There is a representation $\rho_{E,\ell}: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\mathbb{F}_\ell)$ that describes the Galois action on the $\ell$-torsion points of $E$. Building on recent work of Rouse, Zureick-Brown, and Zywina, we discuss composite level modular curves whose rational points classify elliptic curves over $\mathbb{Q}$ with simultaneously non-surjective, composite image of Galois. We shall discuss the techniques used to provably find the rational points on these curves. Finally, we give an application of our results to the study of entanglement fields and answer a question posed by Brau and Jones in a recent paper concerning non-abelian entanglement fields.