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**Colour Critical Hypergraphs With Many Edges**

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# COLOUR CRITICAL HYPERGRAPHS WITH MANY EDGES

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ABSTRACT. We show that for all  $k \geq 3, r > l \geq 2$  there exists constant  $c = c(k, r, l)$  such that for large enough  $n$  there exists a  $k$ -colour-critical  $r$ -uniform hypergraph on less than  $n$  vertices, having more than  $cn^l$  edges, and having no  $l$ -set of vertices occurring in more than one edge.

## 1. INTRODUCTION

A *hypergraph*  $H$  consists of a set  $V = V(H)$  of vertices and a set  $E = E(H)$  of subsets of  $V$ , called edges.

$H$  is  *$k$ -colourable* if  $V$  can be partitioned into  $k$  mutually disjoint colour classes such that no edge  $e \in E$  has all vertices in a single class. A mapping  $\phi : V \rightarrow [k]$  that induces this partition is called a (*proper*)  *$k$ -colouring* of  $H$ , or of  $V(H)$ .

$H$  is  *$k$ -chromatic* if  $k$  is the minimum  $t$  for which  $H$  is  $t$ -colourable.

$H$  is  *$k$ -critical* if it is  $k$ -chromatic but  $H - e$  is  $(k - 1)$ -colourable for any edge  $e \in H$ . Note that we do not allow a  $k$ -critical graph to have isolated vertices.

$H$  is  *$r$ -uniform* if every edge is an  $r$ -subset (subset of size  $r$ ) of  $V$ , and is an  *$(r, l)$ -system* if it is  $r$ -uniform and no  $l$ -subset of  $V$  occurs in more than one edge of  $E$ . (A 2-uniform hypergraph is called a *graph*.)

Recall that the set of 3-critical 2-uniform graphs is the set of odd cycles; however, characterising other sets of critical graphs is a much harder task. The problem of determining if a graph is  $k$ -critical is  $D^P$ -complete [CM87]. Dirac, in [D52] showed for  $k \geq 6$ , and Toft, in [T70], showed for  $k \geq 4$ , that there is a constant  $c = c(k)$  such that for every  $n$  large enough, there is a  $k$ -critical graph on  $n$  vertices with  $\geq cn^2$  edges. In fact, Toft showed that  $c(4) = \frac{1}{16}$ . In [T73], Toft showed (attributing the result to Erdős for  $k = 3$ ) that the maximum number of edges of a  $k$ -critical  $r$ -uniform hypergraph is of order less than  $n^r$  for  $k = 3$  and is of order  $n^r$  for  $k > 3$ . Lovász [L76] improved the upper bound for the maximum number of edges of a 3-critical  $r$ -uniform hypergraph to  $\binom{n}{r-1}$ .

The number of edges in  $k$ -chromatic  $(r, l)$ -systems has been studied elsewhere. In [E75], Erdős and Lovász find bounds for the minimum number of edges that a  $k$ -chromatic  $(r, 2)$ -system can have. In [K01], some of these results are improved and extended to  $(r, l)$ -systems.

The paper [A80] gives a lower bound for the number of non-isomorphic  $k$ -critical  $(r, 2)$ -systems.

In this paper, we extend Toft's results above about the maximum number of edges of  $k$ -critical  $r$ -uniform hypergraphs [T73], to  $(r, l)$ -systems. We also improve the lower bound from [A80] about the number of non-isomorphic  $k$ -critical  $(r, 2)$ -systems, and extend this result to  $(r, l)$ -systems.

## 2. RESULTS

The main result of this paper is the following theorem:

**Theorem 2.1.** *For all  $k \geq 3$  and  $r > l \geq 2$ , there exists constants  $c = c(k, r, l)$  and  $N = N(k, r, l)$  such that, for all  $n \geq N$ , there exists a  $k$ -critical  $(r, l)$ -system on  $n$  vertices, with at least  $cn^l$  edges.*

Notice that up to a constant factor, this is the best possible, since an  $(r, l)$ -system on  $n$  vertices can have at most  $\frac{\binom{n}{l}}{\binom{n}{r}} = \Theta(n^l)$  edges.

We denote by  $S(k, r, l)$  the following statement.

$S(k, r, l)$ : There exist constants  $c = c(k, r, l)$  and  $N = N(k, r, l)$  such that for all  $n \geq N$ , there exists a  $k$ -critical  $(r, l)$ -system  $H$  on  $n$  vertices with  $\geq cn^l$  edges.

Theorem 2.1 can now be restated as follows.

**Theorem 2.1'.** For all  $k \geq 3$  and  $r > l \geq 2$ ,  $S(k, r, l)$  is true.

The following weaker version of statement  $S(k, r, l)$  is easier to prove.

$W(k, r, l)$ : There exist constants  $c_w = c_w(k, r, l)$  and  $N_w = N_w(k, r, l)$  such that for all  $n \geq N_w$  there exists a  $k$ -critical  $(r, l)$ -system  $H$  on  $\nu \leq n$  vertices with  $\geq c_w n^l$  edges.

Note that  $\nu > c_w^{1/l} n$  in this definition, for if it were smaller, the system couldn't have  $\geq c_w n^l$  edges.

It is clear that  $S(k, r, l)$  implies  $W(k, r, l)$ . The inverse implication, however, is more complicated. (Note that we do not allow isolated vertices in  $k$ -critical systems.) We prove this inverse implication in Section 4 with the following lemma.

**Lemma 2.1.**  $W(k, r, l) \Rightarrow S(k, r, l)$  for  $k \geq 3, r > l \geq 2$ .

In view of Lemma 2.1, Theorem 2.1' follows by induction from the following Lemmas:

**Lemma 2.2.**  $W(k, r, l) \Rightarrow W(k, r + 1, l)$  for  $k \geq 3$  and  $r > l \geq 2$

**Lemma 2.3.**  $W(4, l, l) \Rightarrow W(3, l + 1, l)$  for  $l \geq 2$

*Proof of Theorem 2.1'.* The proof is by induction. Since Toft [T73] gives us that for  $k > 3$   $W(k, l, l)$  is true for all  $l > 2$  while Erdős [E73] gives us that  $W(k, l, l)$  is false for  $k = 3$ , we have two different cases.

For  $k > 3$ ,  $W(k, r, l)$  follows immediately from  $W(k, l, l)$  by Lemma 2.2.

For  $k = 3$ , we apply Lemma 2.3 to  $W(4, l, l)$  to get  $W(3, l + 1, l)$ , and then apply Lemma 2.2 to get  $W(3, r, l)$ . □

Section 3 gives some preliminary results, Section 4 gives the proof of Lemma 2.1, Section 5 gives the proofs of Lemmas 2.2 and 2.3, and Section 6 gives a result that follows from Theorem 2.1.

## 3. PRELIMINARY RESULTS

The following will be useful in proving the lemmas.

**Construction 3.1.** For all  $k \geq 2$  and  $r \geq 3$ , we have the following hypergraphs:

- (i)  $k$ -chromatic  $(r, 2)$ -system  $S(k, r)$ , with designated vertices  $x$  and  $y$ , that is edge critical with respect to the property of being  $k$ -colourable iff  $x$  and  $y$  are the same colour;
- (ii)  $k$ -chromatic  $(r, 2)$ -system  $NS(k, r, s)$ , for  $2 \leq s < r$ , with designated vertices  $W = \{w_1, \dots, w_s\}$ , that is edge critical with respect to the property of being  $k$ -colourable iff the vertices of  $W$  are not all the same colour; and
- (iii)  $k$ -chromatic  $(r, 2)$ -system  $D(k, r)$ , with designated vertices  $U = \{u_1, \dots, u_k\}$ , that is edge critical with respect to the property of being  $k$ -colourable iff all vertices of  $U$  are distinct colours.

*Proof.*

- (i) Let  $S'$  be obtained from a  $(k+1)$ -critical  $(r, 2)$ -system by removing edge  $e$  containing vertices  $x$  and  $y$ . Then  $S'$  is  $k$ -colourable with  $\chi(x) = \chi(y)$  under any  $k$ -colouring  $\chi$  of  $S'$ . Remove edges from  $S'$  until it is critical with respect to this property. This new critical system, is the  $S(k, r)$  we desire.

Note that because  $S'$  was an  $(r, 2)$ -system and we removed the edge containing  $x$  and  $y$ , these vertices do not occur together in any edge of  $S$ .

- (ii) Construct  $(r, 2)$ -system  $NS = NS(k, r, s)$  as follows.
  - For  $i = 1, \dots, r-s$ , let  $S_i$  be copies of  $S(k, r)$ , and  $x_i$  and  $y_i$  be the designated vertices in  $S_i$ .
  - For  $i = 2, \dots, r-s$ , identify vertices  $x_i$  and  $y_{i-1}$ .
  - Add edge  $e = \{x_1, x_2, \dots, x_{r-s}, y_{r-s}(=w_1), w_2, \dots, w_s\}$ , where  $w_2, \dots, w_s$  are new vertices.

Now under any  $k$ -colouring of  $NS$ , vertices  $x_1, x_2, \dots, x_{r-s}$ , and  $y_{r-s} = w_1$  must all be the same colour, so because of edge  $e$ ,  $W = \{w_1, \dots, w_s\}$  must not be monochromatic. It is easy to see that  $NS$  is critical with respect to this property.

Note that we use the fact that  $x$  and  $y$  do not occur together in any edge of  $S(k, r)$  to get that  $NS$  is an  $(r, 2)$ -system

- (iii) Construct  $(r, 2)$ -system  $D = D(k, r)$  as follows.
  - Let  $U = \{u_1, \dots, u_k\}$  be a set of vertices.
  - For each pair of vertices  $u_i, u_j \in U$ , take a copy of  $NS(k, r, 2)$ ; identify  $u_i$  with one of its designated vertices, and  $u_j$  with the other.

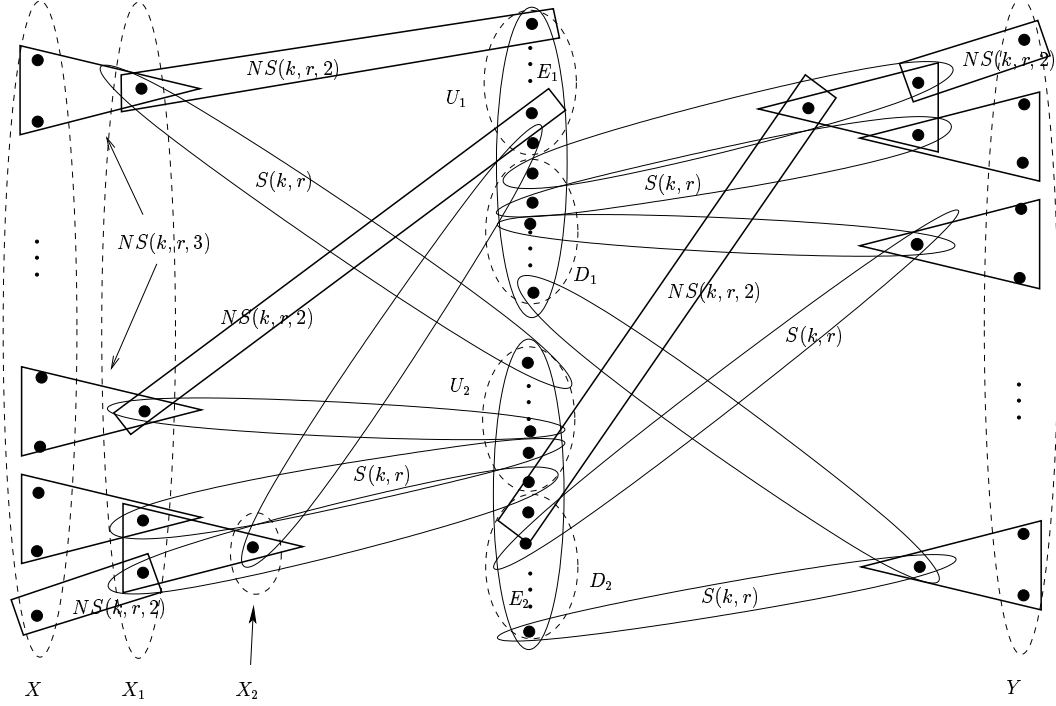
Under any  $k$ -colouring of  $D$ , the vertices of  $U$  all have different colours. It is easy to see that  $D$  is critical with respect to this property. □

#### 4. PROOF OF LEMMA 2.1

We prove Lemma 2.1 with a construction (4.1) that allows us to add any number of vertices greater than some constant, while maintaining  $k$ -criticality.

The following theorem follows from [GL74, Thm 2], which says that the only  $k$ -colourings of  $(K_n)^k$  are the projections onto a single component, and is proved explicitly in a stronger form in [M79, Thm 1].

**Theorem 4.1.** *Let  $k \geq 3$  and  $p > 0$  be integers, and  $\mathbb{A} = \{\mathcal{A}_1, \dots, \mathcal{A}_p\}$  be different partitions of the finite set  $A$  into at most  $k$  classes. Then there exists a  $k$ -chromatic graph  $G$  such that  $A \subset V(G)$  and the set of possible colourings of  $G$ , when restricted to  $A$ , is the set  $\mathbb{A}$ .*

FIGURE 1.  $k = 2$ ,  $r$  odd

We use this theorem in the proof of the following lemma, which is in turn used in Construction 4.1.

**Lemma 4.1.** *Let  $k \geq 2, r \geq 3$  be integers. Then there exists a  $k$ -chromatic  $(r, 2)$ -system  $F$  with  $X = \{x_1, \dots, x_r\} \subset V(F)$  and  $Y = \{y_1, \dots, y_r\} \subset V(F)$  that is critical with respect to the property that  $F$  can be  $k$ -coloured iff  $X$  and  $Y$  are not both monochromatic.*

*Proof.* For  $k = 2$  the proof is by a direct construction, for  $k \geq 3$  it is by a construction based on Theorem 4.1.

Assuming  $k = 2$ ,  $F$  is the system seen in Figures 1 ( $r$  odd), and 2 ( $r$  even). In these figures, the triangles are copies of the system  $NS(k, r, 3)$  except when  $r = 3$ , in which case they are just edges. There are two special edges  $E_1$  and  $E_2$ , and otherwise, the solid ovals are copies of  $S(k, r)$  and the rectangles are copies of  $NS(k, r, 2)$ . The dotted ovals are to apply a label to the sets of vertices shown inside them, either explicitly or implied by ellipses.

We need to show that  $F$  can not be 2-coloured with both  $X$  and  $Y$  monochromatic, and that any other 2-colouring of  $X$  and  $Y$  extends to a 2-colouring of  $F$ . We show the former for  $r$  odd, the latter for  $r$  even, and leave the other two cases to the diligent reader.

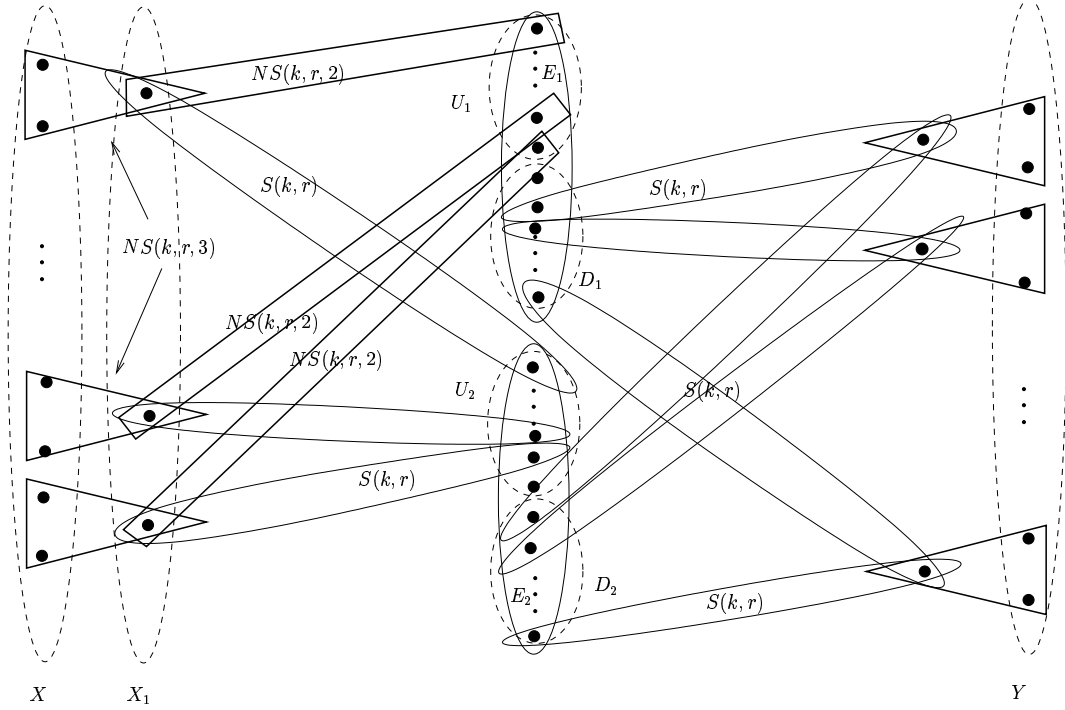


FIGURE 2.  $k = 2$ ,  $r$  even

- $F$  can not be 2-coloured with both  $X$  and  $Y$  monochromatic ( $r$  odd):  
 Assume  $X$  is coloured red. Then  $X_1$  must be blue, and  $X_2$  must in turn be red. This forces  $U_1$  to be red, and  $U_2$  to be blue. Similarly, if  $Y$  is coloured red (blue), then  $D_1$  and  $D_2$  must both be blue (red). Either way, one of  $E_1$  and  $E_2$  is then monochromatic.
- Any other 2-colouring of  $X$  and  $Y$  extends to a 2-colouring of  $F$  ( $r$  even):  
 We show that any 2-colouring of  $X$  other than the monochromatic one extends to a 2-colouring of  $F$ . The proof for  $Y$  is nearly symmetric, so is omitted. Assume  $X$  is coloured with vertices  $x_a$  and  $x_b$  different colours. Let  $NS_a$  and  $NS_b$  be the copies of  $NS(k, r, 3)$  containing  $x_a$  and  $x_b$ , and let  $v_a$  and  $v_b$  be the vertices of  $X_1$  in  $NS_a$  and  $NS_b$  respectively. If  $NS_a = NS_b$ , then  $v_a = v_b$  can be coloured arbitrarily. If  $NS_a \neq NS_b$ , then  $v_a$  and  $v_b$  can be coloured different colours. Either way,  $X_1$  can be coloured such that it has both colours. This forces both  $U_1$  and  $U_2$  to be coloured such that they have both colours. This allows a proper colouring of  $F$  even if  $Y$  is coloured monochromatically.

This finishes the proof for the case  $k = 2$ . For  $k \geq 3$ , let  $A = X \cup Y$  and  $\mathbb{A}$  be all partitions of  $A$  into at most  $k$  classes, in which not both of  $X$  and  $Y$  occur entirely

within their own respective classes. Then Theorem 4.1 gives us  $k$ -chromatic graph  $G$  with  $A \subset V(G)$  whose set of  $k$ -colourings restricted to  $A$  are exactly those by which not both of  $X$  and  $Y$  are monochromatic.

We now make  $G$  into an  $(r, 2)$ -system. For each edge  $e = \{x, y\} \in E(G)$ ,

- let  $NS^e$  be a copy of system  $NS(k, r, 2)$  from Construction 3.1,
- let  $w_1^e$  and  $w_2^e$  be the copies in  $NS^e$  of  $w_1$  and  $w_2$  respectively,
- identify  $x$  with  $w_1^e$  and  $y$  with  $w_2^e$ , and
- remove edge  $e$ .

Call the resulting  $(r, 2)$ -system  $F$ . We observe the following properties of  $F$ .

- $F$  can be properly  $k$ -coloured: Let  $\chi$  be a  $k$ -colouring of  $G$ , for any  $e = \{x, y\}$  in  $E(G)$ ,  $\chi(w_1^e) = \chi(x) \neq \chi(y) = \chi(w_2^e)$  so by the properties of  $NS(k, r, 2)$ ,  $\chi$  can be properly extended to a  $(k)$ -colouring of  $NS^e$ .
- $F$  is not  $(k-1)$ -chromatic:  $F$  contains copies of  $NS(k, r, 2)$ , which cannot be  $(k-1)$ -coloured.
- Any  $k$ -colouring of  $G$  extends to a  $k$ -colouring of  $F$ : We showed this in showing that  $F$  can be properly  $k$ -coloured.
- Any  $k$ -colouring of  $F$  induces a  $k$ -colouring of  $G$ : Let  $\chi$  be a  $k$ -colouring of  $F$ , for any  $e = \{x, y\}$  in  $E(G)$ ,  $\chi(x) = \chi(w_1^e) \neq \chi(w_2^e) = \chi(y)$  by the properties of  $NS(k, r, 2)$ , so  $\chi$  is a  $k$ -colouring of  $G$ .

We have shown that  $F$  is a  $k$ -chromatic  $(r, l)$ -system with  $A = X \cup Y \subset V(J)$  whose  $k$ -colourings restricted to  $A$  are exactly the colourings of  $G$  restricted to  $A$ . This proves the lemma. □

**Theorem 4.2.** [A78, Thm 1] *For  $k, r \geq 3$ , there exists an integer  $M(k, r)$  such that for  $m \geq M(k, r)$  there exists a  $k$ -critical  $(r, 2)$ -system  $K$  on exactly  $m$  vertices.*

The proof of Lemma 2.1 will use Theorem 4.2, and will depend on the following construction.

**Construction 4.1.** *Let  $k \geq 3, r > l \geq 2$ . Given the following systems:*

- $H'$ : a  $k$ -critical  $(r, l)$ -system on  $n'$  vertices,
- $K$ : a  $k$ -critical  $(r, 2)$ -system on  $m$  vertices,
- $F$ : the  $(k-1)$ -chromatic  $(r, 2)$ -system from Lemma 4.1 whose  $(k-1)$ -colourings restricted to  $A = X \cup Y \subset V(F)$  are exactly those that are monochromatic on at most one of  $X = \{x_1, \dots, x_r\}$  and  $Y = \{y_1, \dots, y_r\}$ . (We may assume that  $F$  is critical with respect to this property.)

*There exists a  $k$ -critical  $(r, l)$ -system  $H$  on  $n = n' + f + m - 2r$  vertices with more than  $|E(H')|$  edges, where  $f = |V(F)|$ .*

*Proof.* Construct  $H$  as follows.

- Remove some edge  $e_X = \{\hat{x}_1, \dots, \hat{x}_r\}$  from  $H'$ .
- Remove some edge  $e_Y = \{\hat{y}_1, \dots, \hat{y}_r\}$  from  $K$ .
- For  $i = 1 \dots r$ , identify  $\hat{x}_i$  with  $x_i$  and  $\hat{y}_i$  with  $y_i$ .

Note that since all edges of the components  $H', F$ , and  $K$  of  $H$  were size  $r$ , all the edges of  $H$  have size  $r$ . Since the only  $l$ -set of vertices that may have gained an edge in the construction were entirely within  $X = e_X$  or  $Y = e_Y$ , and edges  $e_X$  and  $e_Y$  were removed,  $H$  is an  $(r, l)$ -system. System  $H$  clearly has  $n = n' + f + m - 2r$  vertices and more than  $|E(H')|$  edges. We now show that it is  $k$ -critical by showing

first that it cannot be  $k-1$ -coloured, and then showing that it can be  $k-1$ -coloured with the removal of any edge.

- Assume  $H$  has  $(k-1)$ -colouring  $\chi$ . By the  $k$ -criticality of  $H'$ , the vertices of  $e_X = X$  are monochromatic. Then by the properties of  $F$ , the vertices of  $Y = e_Y$  are not monochromatic. However, since  $K$  was  $k$ -critical,  $K - e_Y$  can be  $(k-1)$ -coloured only if the vertices of  $e_Y$  are monochromatic. This is a contradiction.
- Remove edge  $e$  from  $H' - e_X$ , then  $H' - \{e_X, e\}$  has a  $(k-1)$ -colouring  $\chi$  under which the vertices of  $e_X = X$  are not monochromatic. Since  $X$  is not monochromatic,  $\chi$  can be extended to a  $(k-1)$ -colouring of  $F$  under which  $Y = e_Y$  is monochromatic. This colouring can then be extended to a  $(k-1)$ -colouring of  $K - e_Y$ .  
The argument for removing edge  $e$  from  $K - e_Y$  is symmetric.
- Remove edge  $e$  from  $F$ . Since  $F$  was critical with respect to the property described in Construction 4.1, it can now be  $(k-1)$ -coloured with both  $X$  and  $Y$  monochromatic. This colouring can then be extended to a  $(k-1)$ -colouring of both  $H' - e_X$  and  $K - e_Y$ .

We have shown that  $H$  can not be  $(k-1)$ -coloured, but can be with the removal of any edge. Thus  $H$  is  $k$ -critical, and the proof is done.  $\square$

*Proof of Lemma 2.1.* The direction  $S(k, r, l) \Rightarrow W(k, r, l)$  is trivial.

For the other direction, assume  $W(k, r, l)$ ; i.e. that we have constants  $c_w = c_w(k, r, l)$  and  $N_w = N_w(k, r, l)$  such that for  $N' \geq N_w$  there exists a  $k$ -critical  $(r, l)$ -system  $H'$  on  $n' \leq N'$  vertices, with more than  $c_w n'^l$  edges.

We will deliver  $S(k, r, l)$  by finding constants  $c$  and  $N$  such that for all  $n \geq N$  there exists a  $k$ -critical  $(r, l)$ -system  $H$  on exactly  $n$  vertices and having at least  $cn^l$  edges.

Let  $F$  be the  $k$ -chromatic  $(r, 2)$ -system from Lemma 4.1, let  $M(k, r)$  be the constant from Theorem 4.2, and set  $c = \frac{c_w}{2^r}$  and  $N > 2 \max(N_w, f + M(k, r))$ . Let  $n \geq N$ . Then  $\lceil n/2 \rceil > N_w$ , so by  $W(k, r, l)$  there exists a  $k$ -critical  $(r, l)$ -system  $H'$  on  $n' \leq \lceil n/2 \rceil$  vertices with more than  $\frac{c_w}{2^r} n'^l = cn^l$  edges.

Since  $n - n' - f + 2r > n/2 - f > M(k, r)$ , Theorem 4.2 gives us a  $k$ -critical  $(r, 2)$ -system  $K$  on exactly  $n - n' - f + 2r$  vertices.

Applying Construction 4.1 to  $H'$  and  $K$  gives us the required  $k$ -critical  $(r, l)$ -system  $H$  on  $n' + f + (n - n' - f + 2r) - 2r = n$  vertices and having at least  $|E(H')| = cn^l$  edges.

We have shown that for any  $n \geq N > 2 \max(N_w, f + M(k, r))$ , there exists a  $k$ -critical  $(r, l)$ -system  $H$  on  $n$ -vertices having at least  $cn^l$  edges. This completes the proof.  $\square$

## 5. PROOFS OF LEMMAS 2.2, AND 2.3

**Lemma 2.2**  $W(k, r, l) \Rightarrow W(k, r + 1, l)$  for  $k \geq 3$  and  $2 \leq l < r$

**Lemma 2.3**  $W(4, l, l) \Rightarrow W(3, l + 1, l)$  for  $l \geq 2$

The proofs of the two lemmas are both generalizations of an old (circa 1980) construction by the senior author which gives the case  $W(4, 2, 2) \Rightarrow W(3, 3, 2)$ . In a personal communication [S84] M. Stiebitz suggested how to generalize the construction to give  $W(k, r, 2) \Rightarrow W(k, r + 1, 2)$ . He also used a construction to



show that  $W(k, r, 2)$  is true for all  $r, k \geq 3$ . The proofs given here are different and broader generalizations. They are simpler than Stiebitz's due to the use of a more recent result of Pippenger and Spencer which deals with the decompositions of regular hypergraphs into matchings. (A *matching* of a hypergraph  $H$  is a subset of its edges such that no vertex occurs in more than one edge of the subset.)

For the proofs of both lemmas we need the aforementioned result of Pippenger and Spencer:

**Theorem 5.1.** [PS87]

*For every  $r \geq 2$  and  $\mu > 0$ , there exist  $\mu' = \mu'(r, \mu) > 0$  and  $m_0 = m_0(r, \mu)$  such that if  $H$  is an  $r$ -uniform  $D$ -regular hypergraph on  $m \geq m_0$  vertices with maximum co-degree  $C \leq \mu'D$ , then the edges of  $H$  can be partitioned into less than  $D(1 + \mu)$  matchings. (The co-degree of a pair of vertices is the number of edges in which they occur together.)*

**Proposition 5.1.** *For large enough  $m$ , the edges of an  $(r, l)$ -system  $H$  on  $m$  vertices can be decomposed into  $t \leq 2 \frac{m-l+1}{r-l+1}$   $(r, l-1)$ -systems  $M_\alpha$ . I.e.  $E(H) = \bigcup_{\alpha=1}^t E(M_\alpha)$ .*

In the proof of Proposition 5.1, we use the following auxillary construction. Form hypergraph  $H'$  from  $H$  as follows.

$$V(H') := \binom{V(H)}{l-1} \quad (\text{i.e. the set of } (l-1)\text{-subsets of } V(H))$$

$$E(H') := \left\{ \binom{e}{l-1} \mid e \in E(H) \right\}$$

By this construction, each edge  $e$  of  $H$  becomes an edge  $e'$  in  $H'$  whose vertices are exactly the  $(l-1)$ -subsets of the vertices in  $e$ .

We note the following properties of  $H'$ :

- Since every edge  $e \in E(H)$  has  $r$  vertices, every edge  $e' \in E(H')$  has  $\binom{r}{l-1}$  vertices. Thus  $H'$  is  $\binom{r}{l-1}$ -uniform.
- Since no  $l$ -subset of  $V(H)$  occurs in more than one edge, any  $(l-1)$ -subset can occur in at most  $\frac{m-(l-1)}{r-(l-1)}$  edges. Thus any vertex of  $H'$  can occur in at most  $\frac{m-(l-1)}{r-(l-1)}$  edges. Consequently,  $H'$  has maximum degree  $D' \leq D = \frac{m-(l-1)}{r-(l-1)}$ .
- Since no  $l$ -subset of  $V(H)$  occurs in more than one edge, no two  $(l-1)$ -subsets of  $V(H)$  occur together in more than one edge. Thus no two vertices of  $H'$  occur together in more than one edge. Consequently,  $H'$  has a maximum co-degree of 1.
- A matching of  $H'$  corresponds to a set of edges of  $H$  no two of which share an  $(l-1)$ -subset of  $V(H)$ . This is exactly an  $(r, l-1)$ -system.

Let  $\delta(G)$  denote the minimum degree of hypergraph  $G$ .

The proof of Proposition 5.1 is based on the following claim.

**Claim.**  $H'$  can be embedded in  $\binom{r}{l-1}$ -uniform  $D$ -regular hypergraph  $H''$  with maximum co-degree 1. (This will allow us to use Theorem 5.1).

*Proof.* We prove this by forming a sequence of hypergraphs  $H' = H_0, H_1, \dots, H_{D-\delta(H')} = H''$ , all of maximum degree  $D$  and maximum co-degree 1, where for all  $i$ ,  $H_i$  is embedded in  $H_{i+1}$ , and  $\delta(H_{i+1}) = \delta(H_i) + 1$ . Given  $H_i$ , we form  $H_{i+1}$  as follows. Take  $r$  copies  $H_i = H_i^1, H_i^2, \dots, H_i^r$  of  $H_i$ , and for every vertex  $v$  in  $V(H_i)$  let  $v^j$  be the copy of  $v$  in  $H_i^j$ . For every vertex  $w$  of minimum degree  $\delta + i$  in  $H_i$  we add the edge  $\{w^1, \dots, w^r\}$ . This increases the degree of every vertex of minimum degree by 1, and leaves the degrees of other vertices unchanged. The only co-degrees affected are those between copies of a minimum degree vertex, and are raised from 0 to 1.  $\square$

*Proof of Proposition 5.1.* Taking  $\mu < 1$  in Theorem 5.1 and  $m > m_0(r, \mu)$  large enough that  $1 < \mu'(r, \mu)D$ , we can apply the theorem to  $H''$  to get a decomposition of its edges into  $t \leq 2D$  matchings. These are still matchings when restricted to  $H'$ . As noted before, this corresponds to a decomposition of  $H$  into  $t$   $(r, l - 1)$ -systems.  $\square$

Now we are ready to prove Lemma 2.2.

*Proof.* Assuming  $W(k, r, l)$ , we have a  $k$ -critical  $(r, l)$ -system  $H$  with  $m$  vertices  $V(H) = \{b_1, \dots, b_m\}$  and  $c(k, r, l)m^l$  edges. We construct  $(r + 1, l)$ -system  $J$  with  $n < m(|V(D(k - 1, r + 1))| + |V(S(k - 1, r + 1))|)$  vertices (where  $D(k - 1, r + 1)$  and  $S(k - 1, r + 1)$  are from Construction 3.1). We then show that any  $k$ -critical subsystem of  $J$  has at least  $|E(H)| = c(k, r, l)m^l$  edges. This will prove  $W(k, r + 1, l)$  with  $c(k, r + 1, l) > \frac{c(k, r, l)}{(|V(D(k - 1, r + 1))| + |V(S(k - 1, r + 1))|)^t}$ .

By Proposition 5.1 there exist  $t \leq 2\frac{m-l+1}{r-l+1}$   $(r, l - 1)$ -systems  $M_\alpha$  such that  $E(H) = \bigcup_{\alpha=1}^t E(M_\alpha)$

We construct system  $J$  as follows:

- For  $\alpha = 1, \dots, t - 1$ , let  $S^\alpha$  be copies of the hypergraph  $S(k - 1, r + 1)$  from Construction 3.1. Let  $x^\alpha$  and  $y^\alpha$  be the corresponding designated vertices.
- For each  $\alpha = 1, \dots, t - 2$ , identify  $x^{\alpha+1}$  and  $y^\alpha$ , and relabel them to obtain  $w_{\alpha+1} = x^{\alpha+1} = y^\alpha$ . Relabel  $x^1$  as  $w_1$  and  $y^{t-1}$  as  $w_t$ . (See Figure 3). Observe that  $W = \{w_1, \dots, w_t\}$  is then monochromatic under any  $(k - 1)$ -colouring of  $\bigcup_{\alpha=1}^{t-1} S^\alpha$ .
- For  $i = 1, \dots, k - 1$ , let  $H^i$  be a copy of  $H$ . For any  $b \in V(H)$ , and  $e \in E(H)$ , let  $b^i$  and  $e^i$  be their respective copies in  $H^i$ . For  $i = 1, \dots, k - 1$  and  $\alpha = 1, \dots, t$ , let  $M_\alpha^i$  be the copy of  $M_\alpha$  in  $H^i$ .
- For  $\alpha = 1, \dots, t$ , every  $e \in M_\alpha$ , and all  $i = 1, \dots, k - 1$ , replace edge  $e^i \in M_\alpha^i$  with edge  $\tilde{e}^i = e^i \cup \{w_\alpha\}$ . Let  $\tilde{M}$  be all edges created in this way. More precisely, let  $\tilde{M} = \bigcup_{i=1, \dots, k-1} \bigcup_{\alpha=1, \dots, t} \tilde{M}_\alpha^i$ , where  $\tilde{M}_\alpha^i = \{\tilde{e}^i | e^i \in M_\alpha^i\}$ .

(See Figure 4). Observe that this replacement of  $r$ -sets with  $(r + 1)$ -sets yields an  $(r + 1)$  uniform hypergraph which, because each  $H^i$  was partitioned into  $(r, l - 1)$ -systems  $M_\alpha$ , is in fact an  $(r + 1, l)$ -system.

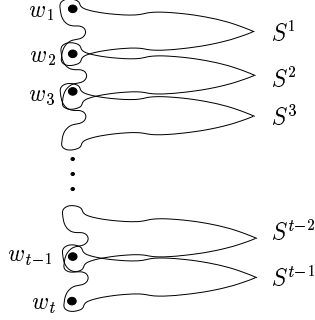
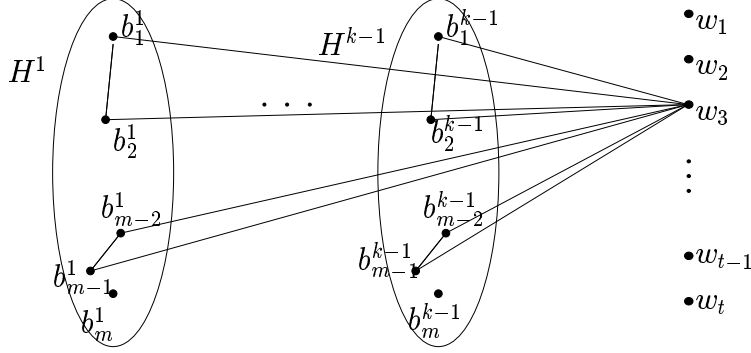


FIGURE 3

FIGURE 4. Edges of  $\widetilde{M}_1^3$  and  $\widetilde{M}_{k-1}^3$  ( $r = 2$ ).

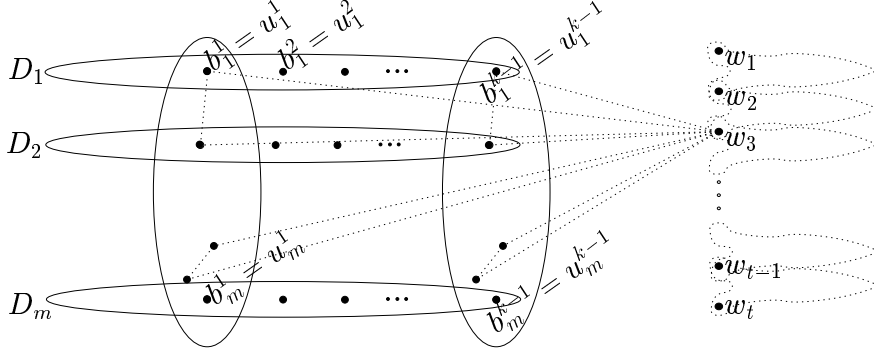
- Let  $D_j$ , for  $j = 1, \dots, m$ , be a copy of  $D(k-1, r+1)$  from Construction 3.1. Let  $U_j = \{u_j^1, \dots, u_j^{k-1}\}$  be the corresponding designated vertices.
- For all  $i = 1, \dots, k-1$ , and  $j = 1, \dots, m$ , identify  $b_j^i$  with  $u_j^i$ . (Compare Figures 4 and 5). Observe that for any  $j = 1 \dots m$ , the vertices  $\{b_j^1, \dots, b_j^{k-1}\}$ , are all different colours under any  $(k-1)$ -colouring of  $J$ .

More precisely,  $J$  is

$$V(J) = \bigcup_{i=1}^{k-1} V(H^i) \cup \bigcup_{\alpha=1}^{t-1} V(S^\alpha) \cup \bigcup_{j=1}^m V(D_j)$$

$$E(J) = \widetilde{M} \cup \bigcup_{\alpha=1}^{t-1} E(S^\alpha) \cup \bigcup_{j=1}^m E(D_j)$$

Observe that  $J$  is an  $(r+1, l)$ -system. It has  $n = (k-1)m + (t-1)|V(S)| + m|V(D) - (k-1)| - (t-2) < m|V(D)| + 2\frac{m}{r}|V(S)| < m(|V(D)| + |V(S)|)$  vertices.

FIGURE 5. Identification of  $b_j^i$  with  $u_j^i$ .

To finish the proof of Lemma 2.2, we have to show that there is a  $k$ -critical subsystem of  $J$  that has at least  $|E(H)|$  edges. We do this with the following two claims.

**Claim.**  $J$  is not  $(k-1)$ -colourable.

*Proof.* Assume  $J$  is properly coloured with  $(k-1)$  colours. Since  $W$  is monochromatic, we may assume it is colour 1. For  $i = 1, \dots, k-1$ , let  $C^i$  be the vertices of  $H^i$  of colour 1, and let  $\tilde{C}^i = \{b_j^i | b_j^i \in C^i\}$ .

For any  $j = 1, \dots, m$ , the vertices  $\{b_j^1, \dots, b_j^{k-1}\}$  are all different colours, so exactly one of them occurs in  $\bigcup_{i=1}^{k-1} C^i$ . Thus the sets  $\tilde{C}^i$ , for  $i = 1, \dots, k-1$  form a  $(k-1)$ -colouring of  $V(H)$ . This contradicts the fact that  $H$  is  $k$ -critical.  $\square$

**Claim.** Let  $K$  be any subgraph of  $J$  that is not  $(k-1)$  colourable. (In particular,  $K$  may be a  $k$ -critical subgraph.) Then for every edge  $e \in E(H)$ ,  $K$  must contain at least one of the edges  $\tilde{e}^1, \dots, \tilde{e}^{k-1}$ . Consequently,  $K$  has at least  $|E(H)|$  edges.

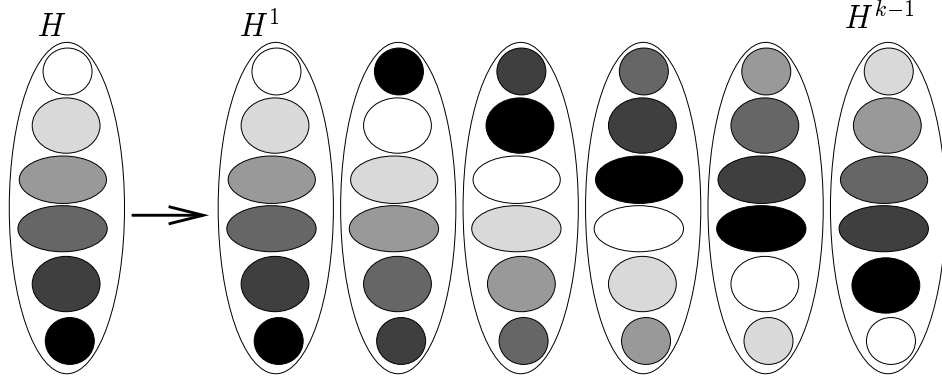
*Proof.* For some  $e \in E(H)$ , assume  $K_e \subset J$  has none of the edges  $\tilde{e}^1, \dots, \tilde{e}^{k-1}$ .  $H$  is  $k$ -critical, so  $H - e$  has proper  $(k-1)$ -colouring  $\bigcup_{\gamma=1}^{k-1} C_\gamma$ . For  $i = 1, \dots, k-1$ , and  $\gamma = 1, \dots, k-1$  let  $C_\gamma^i$  be the copy of  $C_\gamma$  in  $H^i$ .

For  $i = 1, \dots, k-1$ , let  $\chi(b) = i$  for all  $b \in \bigcup_{\gamma=1}^{k-1} C_\gamma^{i+1}$  where the upper index is taken modulo  $k-1$ . (See Figure 6).

It remains to show that  $\chi$  extends to a proper  $(k-1)$ -colouring of  $K_e$ .

For any  $i = 1, \dots, k-1$ ,  $\alpha = 1, \dots, t$ , and any edge  $\tilde{d}^i = d^i \cup \{w_\alpha\} \in \widetilde{M}_\alpha^i$ ,  $d^i$  already has more than one colour under  $\chi$ . This is because  $d^i \in V(H^i)$ , and  $\bigcup_{\gamma=1}^{k-1} C_\gamma^i$  is a proper colouring of  $H^i$ .

Since  $d^i$  already has more than one colour under  $\chi$ , we may arbitrarily define  $\chi$  on  $w_\alpha$ . Let  $\chi(w_\alpha) = 1$  for  $\alpha = 1, \dots, t$ .

FIGURE 6. Definition of  $\chi$  ( $k=7$ )

By the properties of  $S(k-1, r+1)$ ,  $\chi$ , which is monochromatic on  $W$ , can be further extended to a proper  $(k-1)$ -colouring of  $\bigcup_{\alpha=1}^{t-1} S^\alpha$ .

Observe that for  $j = 1, \dots, m$ ,  $\{\chi(b_j^1), \dots, \chi(b_j^{k-1})\} = \{1, \dots, k-1\}$ . Thus by the properties of  $D(k-1, r+1)$ ,  $\chi$  can be extended again to a proper  $(k-1)$ -colouring of  $\bigcup_{j=1}^m D^j$ .

Consequently,  $K_e$  can be  $(k-1)$ -coloured, proving by contradiction that for every edge  $e \in E(H)$ , a  $k$ -critical subsystem of  $J$  must contain at least one of the edges  $\tilde{e}^1, \dots, \tilde{e}^{k-1}$ .  $\square$

We have shown that  $(r+1, l)$ -system  $J$ , which has  $n$  vertices, is a  $k$ -critical subsystem that has at least  $|E(H)|$  edges. By the discussion at the beginning of the proof, this completes the proof of the lemma.  $\square$

The proof of Lemma 2.3 is similar to that of Lemma 2.2, but the differences require some exposition.

*Proof.* We begin with essentially the same construction we used in Lemma 2.2. From 4-critical  $(l, l)$ -system  $H$ , we build  $(l+1, l)$ -system  $J$  exactly as we did in the proof of Lemma 2.2 only we use copies of  $NS(2, l+1, 3)$  instead of copies of  $D(3, l+1)$ .

Explicitly, start with 4-critical  $(l, l)$ -system  $H$  with  $m$  vertices and  $cm^l$  edges. By Proposition 5.1, the edges can be decomposed into  $t < m$   $(l, l-1)$ -systems. Take three copies of  $H$  and  $t-1$  copies of  $S(2, l+1)$ . Identify their vertices as in Lemma 2.2. Take  $m$  copies of  $NS(2, l+1, 3)$ , and attach them to the copies of  $H$  in the same way the copies of  $D(k-1, l+1)$  were attached to  $H$  in the proof of Lemma 2.2. Again, call this new system  $J$ . (See Figure 7).

**Claim.**  $J$  is not 2-colourable.

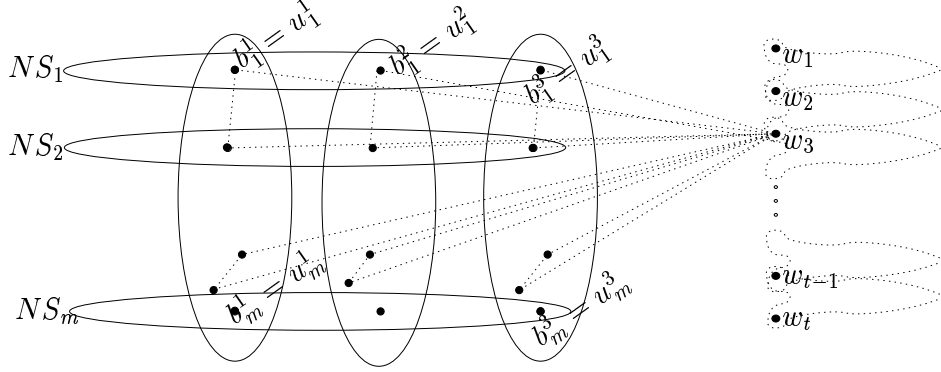


FIGURE 7

*Proof.* Assume  $J$  is properly coloured with 2 colours. Since  $W$  is monochromatic, we may assume it is colour 1. For  $i = 1, 2, 3$ , let  $C^i$  be the set of vertices of  $H^i$  of colour 1, and let  $\tilde{C}^i = \{b_j | b_j^i \in C^i\} \subset V(H)$ .

For any  $j = 1, \dots, m$ , the vertices  $\{b_j^1, b_j^2, b_j^3\}$  are not all the same colour by the properties of  $NS^j$ , so at least one of them occurs in  $\bigcup_{i=1}^3 C^i$ . Thus the union of the sets  $\tilde{C}^i$  for  $i = 1, 2, 3$  contain all of  $V(H)$ . Observe that for  $i = 1, 2, 3$ ,  $\tilde{C}^i$  is independent. This contradicts the fact that  $H$  is 4-critical.  $\square$

**Claim.** Let  $K$  be any subgraph of  $J$  that is not 2 colourable. (In particular,  $K$  may be a 3-critical subgraph.) Then for every edge  $e \in E(H)$ ,  $K$  must contain at least one of the edges  $\tilde{e}^1, \tilde{e}^2, \tilde{e}^3$ . Consequently,  $K$  has at least  $|E(H)|$  edges.

*Proof.* For some  $e \in E(H)$ , assume  $K_e \subset J$  has none of the edges  $\tilde{e}^1, \tilde{e}^2, \tilde{e}^3$ . Since  $H$  is 4-critical,  $H - e$  has proper 3-colouring  $\bigcup_{\gamma=1}^3 C_\gamma$ . For  $i = 1, 2, 3$ , and  $\gamma = 1, 2, 3$

let  $C_\gamma^i$  be the copy of  $C_\gamma$  in  $H^i$ .

Let  $\chi(v) = 1$  for all  $v \in C_1^1 \cup C_2^2 \cup C_3^3 \cup W$ , and let  $\chi(v) = 2$  for all other  $v$  in  $V(H^1 \cup H^2 \cup H^3)$ .

It remains to show that  $\chi$  extends to a proper 2-colouring of  $K_e$ .

For any  $i = 1, 2, 3$ ,  $\alpha = 1, \dots, t$ , and any edge  $\tilde{d}^i = d^i \cup \{w_\alpha\} \in \tilde{M}_\alpha^i$ , vertex  $w_\alpha$  is coloured 1 by  $\chi$  and edge  $\tilde{d}^i$  contains  $l$  more vertices from  $H^i$ . These  $l$  vertices are all in edge  $d^i$  of  $H^i$ , so at least one of them is not from  $C_\alpha^i$ . This one is coloured 2 by  $\chi$ .

By the properties of  $S(2, l+1)$ ,  $\chi$ , which is monochromatic on  $W$ , can be further extended to a proper 2-colouring of  $\bigcup_{\alpha=1}^{t-1} S^\alpha$ .

Observe that for  $j = 1, 2, 3$ ,  $\{\chi(b_j^1), \chi(b_j^2), \chi(b_j^3)\} = \{1, 2\}$ . Thus by the properties of  $NS(2, l+1, 3)$ ,  $\chi$  can be extended again to a proper 2-colouring of  $\bigcup_{j=1}^m NS^j$ .

Consequently,  $K_e$  can be 2-coloured, proving by contradiction that for every edge  $e \in E(H)$ , a  $k$ -critical subsystem of  $J$  must contain at least one of the edges  $\tilde{e}^1, \tilde{e}^2, \tilde{e}^3$ .  $\square$

We have shown that  $(l+1, l)$ -system  $J$ , has a  $k$ -critical subsystem which has  $n < m(|V(NS(2, l+1, 3))| + |V(S(2, l+1))|)$  vertices and at least  $|E(H)|$  edges. This completes the proof of the lemma with  $c(3, l+1, l) > \frac{c(4, l, l)}{|V(NS(2, l+1, 3))| + |V(S(2, l+1))|}$ .  $\square$

## 6. CONSEQUENTIAL RESULTS

Let  $T(k, r, l, n)$  be the number of non-isomorphic  $k$ -critical  $(r, l)$ -systems on  $n$  vertices. It was shown in [A80] that for all  $r, k \geq 3$ , there exists constant  $d = d(r, k, 2) > 1$  such that  $T(k, r, 2, n) > d^n$  for large enough  $n$ . We can now improve this result, and extend it to all  $2 \leq l < r$ .

**Theorem 6.1.** *For all  $k \geq 3$  and  $2 \leq l < r$ , there exists  $d = d(k, r, l)$  and  $n_0 = n_0(k, r, l)$  such that for  $n > n_0$ ,  $T(k, r, l, n) > d^{n^l}$*

*Proof.* For  $k, r, l$  fixed and  $n > n_0$ , we will

- (i) Choose appropriate  $n^* < n$ .
- (ii) Use Theorem 2.1 to get a  $k$ -critical  $(r, l)$ -system  $H^*$  on  $n^*$  vertices with  $\geq c(n^*)^l$  edges.
- (iii) For some  $a = a(k, r, l) > 1$  and  $s = s(k, r, l) > 1$ , construct more than  $a^{n^*}$  distinct  $k$ -critical  $(r, l)$ -systems  $H'$  each on between  $n^*$  and  $s \cdot n^*$  vertices.
- (iv) Apply Construction 4.1 to each of the  $H'$  to get more than  $a^{(n^*)^l}$  distinct  $k$ -critical  $(r, l)$ -systems  $H$  each on exactly  $n$  vertices.
- (v) Show that for some  $d = d(k, r, l) > 1$  at least  $d^{n^l}$  of these systems are non-isomorphic.

This will prove the theorem.

(i) Let  $N = N(k, r, l)$ ,  $f$ ,  $M(k, r)$ , and  $S = S(k-1, r)$  be from Theorem 2.1, Construction 4.1, Theorem 4.2, and Construction 3.1 respectively. Let

$$n_0 > 2 \max\{N|V(S)|, f + M(k, r) + |V(S)|\} \quad (1)$$

and let  $n > n_0$ . Choose  $n^*$  to be the maximum integer such that

$$n > n^*|V(S)| + f + M(k, r) \geq N|V(S)| + f + M(k, r). \quad (2)$$

(ii) By choice of  $n^*$ , we have that

$$n^* \geq N. \quad (3)$$

So by Theorem 2.1 we have a  $k$ -critical  $(r, l)$ -system  $H^*$  on  $n^*$  vertices with  $\geq c(n^*)^l$  edges.

Later, we will also use that

$$\frac{n}{n^*} < 2|V(S)|. \quad (4)$$

Indeed by (1),

$$(n^* + 1)|V(S)| + f + M(k, r) > n > 2(f + M(k, r) + |V(S)|),$$

so

$$(n^* - 1)|V(S)| > f + M(k, r),$$

consequently,

$$2n^*|V(S)| \geq (n^* + 1)|V(S)| + f + M(k, r) \geq n,$$

which, implies (4).

(iii) For any of the  $2^{|E(H^*)|} > 2^{c(n^*)^l}$  possible partitions  $E_1 \cup E_2$  of  $E(H^*)$ , we construct the following system  $H''$ . ( $H'$  will be a critical subsystem of  $H''$ .)

**Construction 6.1.** Set  $V = V(H^*) = \{v_1, \dots, v_{n^*}\}$ , and let  $E_1 \cup E_2$  be any partition of the edges of  $H^*$ . For  $j = 1, 2$  let  $V^j = \{v_1^j, \dots, v_{n^*}^j\}$  be vertex disjoint copies of  $V$  and let

$$E'_j = \{\{v_{i_1}^j, \dots, v_{i_k}^j\} \mid \{v_{i_1}, \dots, v_{i_k}\} \in E_j\},$$

For  $i = 1, \dots, n^*$ , let  $S_i$  be a copy of  $S(k-1, r)$  from Construction 3.1. Let  $w_i^1$  and  $w_i^2$  be the copies of  $x$  and  $y$  in  $S_i$ , and identify  $v_i^1$  and  $v_i^2$  with  $w_i^1$  and  $w_i^2$  respectively.

Define  $H'' = H''(E_1 \cup E_2)$  as follows:

$$V(H'') = V^1 \cup V^2 \cup \bigcup V(S_i)$$

$$E(H'') = E'_1 \cup E'_2 \cup \bigcup E(S_i).$$

For each partition  $(E_1 \cup E_2)$  of  $E(H^*)$  we have constructed an  $(r, l)$  system  $H''$  such that  $n'' = |V(H'')| = |V(H^*)| \cdot |V(S)| = n^*|V(S)|$ . The following claim implies that  $H''$  has a  $k$ -critical sub-system  $H'$  with at least  $n^*$  vertices, and that the  $H'$  constructed for two different partitions of  $E(H^*)$  are distinct.

**Claim 6.1.** *The  $(r, l)$ -system  $H''$  is not  $(k-1)$ -colourable; moreover, any  $k$ -critical subsystem of  $H'$  of  $H'' = H''(E_1 \cup E_2)$  contains all edges of  $E'_1 \cup E'_2$ .*

*Proof.* Assume that  $H''$  has a proper  $(k-1)$ -colouring  $\chi''$ . Then for all  $i = 1, \dots, n''$ ,  $\chi''(v_i^1) = \chi''(v_i^2)$ . This is because  $v_i^1$  and  $v_i^2$  are identified with  $w_i^1$  and  $w_i^2$  respectively, and the latter two are the same colour by the properties of  $S_i$ . The colouring  $\chi^*$  defined by  $\chi^*(v_i) = \chi''(v_i^1) = \chi''(v_i^2)$  is thus a well defined mapping from  $V(H^*)$  to  $[k-1]$ . For all  $e \in E(H^*)$ , since  $\chi''$  properly colours  $e^1$  or  $e^2$  (whichever exists),  $\chi^*$  properly colours  $e$ . Consequently,  $\chi^*$  is a proper  $(k-1)$ -colouring of  $H^*$ , which contradicts  $H^*$  being  $k$ -critical. Thus we conclude that  $H''$  is not  $(k-1)$ -colourable.

To see that any  $k$ -critical subsystem  $H'$  of  $H''$  contains all edges of  $E'_1 \cup E'_2$ , we show that the removal of any such edge yields a proper  $(k-1)$ -colouring  $\chi'$ .

Remove edge  $e' \in E'_1 \cup E'_2$  from  $H'$ . Without loss of generality, assume  $e' = e^1 = \{v_1^1, \dots, v_k^1\} \in E'_1$  where  $e = \{v_1, \dots, v_k\} \in E$ . By the criticality of  $H^*$ , there exists  $(k-1)$ -colouring  $\chi^*$  of  $H^* - e$ . For  $i = 1 \dots n^*$ , let  $\chi'(v_i^1) = \chi'(v_i^2) = \chi(v_i)$ ; by the identification of  $v_i^1$  and  $v_i^2$  with  $w_i^1$  and  $w_i^2$ , we then get  $\chi'(w_i^1) = \chi'(w_i^2)$ . By the properties of  $S_i$ ,  $\chi'$  can then be properly extended to a  $(k-1)$ -colouring of  $V(S_i)$ . This is true for each  $i = 1 \dots n^*$ , so  $\chi'$  extends to a proper  $(k-1)$ -colouring of  $V(H')$ . □

We have thus constructed  $2^{c(n^*)^l}$  distinct  $k$ -critical  $(r, l)$ -systems  $H'$  with  $n^* \leq |V(H')| \leq n^*|V(S(k-1, r))|$ .



(iv) For each of these  $H'$ , let  $m = n - |V(H')| - f + 2r \geq n - n^*|V(S)| - f + 2r > M(k, r)$ . Then apply Construction 4.1 to  $H'$  and the system  $K$  on  $m$  vertices that we get from Theorem 4.2. This gives us a  $k$ -critical  $(r, l)$ -system  $H$  on exactly  $n$  vertices. We have thus constructed  $2^{c(n^*)^l} = 2^{\frac{c(n^*)^l}{n^l} n^l} > 2^{\frac{c}{2^{|V(S)|l}} n^l} = 2^{d'' n^l}$  distinct such  $H$ , where  $d'' = \frac{c}{2^{|V(S)|l}}$ , and the inequality is by (4).

(v) The isomorphism class of any system on  $n$ -vertices has size at most  $n!$ , so we have at least  $\frac{2^{d'' n^l}}{n!} = 2^{d'' n^l(1+o(1))} > 2^{d' n^l}$  non-isomorphic  $k$ -critical  $(r, l)$ -systems on  $n$  vertices for some constant  $d'$ . This is  $d^{n^l}$  where  $d = 2^{d'}$ .  $\square$

Notice that in the construction of the more than  $d^{n^l}$  different  $k$ -critical  $(r, l)$ -systems, all of the systems had more than  $c' n^l := c(\frac{n^*}{n})^l n^l$  edges.

A trivial upper bound for the number of  $(r, l)$ -systems with  $c' n^l$  edges is  $\binom{\binom{n}{r}}{c' n^l} < \binom{n^r}{c' n^l} \approx (n^{r-l})^{n^l} = O(d^{n^l \log n^{r-l}})$ . We conjecture that the actual number is in fact exponential in  $n^l$ .

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