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**Spanning trees of small degree**

by

Domingos Dellamonica Jr.

Mathematics and Computer Science

EMORY UNIVERSITY

# SPANNING TREES OF SMALL DEGREE

DOMINGOS DELLAMONICA JR.

Abstract. In this paper we show that pseudo-random graphs contain spanning trees of maximum degree 3. More specifically, $(n, d, \lambda)$-graphs with sufficiently large spectral gap contain such spanning trees.

## 1. Introduction

A classical problem in graph theory consists in determining sufficient conditions for a graph to be Hamiltonian. Even though many techniques were developed and many results exist, it is often difficult to prove Hamiltonicity even for well-structured classes of graphs.

For the binomial random graph model $G_{n,p}$, where each possible edge in $\{1, \ldots, n\}$ is selected with probability $p$, the threshold for the Hamiltonicity property was found after a breakthrough by Pósa [Pós76] (see also Bollobás book on Random Graphs [Bol01]). An alternative way of seeing $G_{n,p}$ is by considering a *graph process* where edges are sequentially added to an empty graph on $n$ vertices: sample a real number in $[0, 1]$ independently for each possible edge; at time $p \in [0, 1]$ all edges that have an associated number $\leq p$ were added by the process. The equivalence with $G_{n,p}$ easily follows.

A more refined analysis of the Hamiltonicity problem for random graphs reveals that the main obstacle is the presence of vertices of degree less than 2. In fact, **a.a.s.** (asymptotically almost surely) as soon as a graph with minimum degree 2 is obtained in the random process, the graph becomes Hamiltonian. The threshold $p \sim \log n/n$ for Hamiltonicity is just high enough so that **a.a.s.** there are no vertices of degree less than 2.

The next natural step is to enforce a minimum degree condition on the random graph, for instance, by asking it to be $d$-regular. In the model $G_{n,d}$, where every $d$-regular graph on $n$ vertices has equal probability, we can ask for the probability of sampling a Hamiltonian graph. It turns out that for $d \geq 3$, a graph in $G_{n,d}$ **a.a.s.** contains a spanning cycle (see Theorem 2.26 of Wormald's survey [Wor99]).

It would be very interesting to extend this result to regular graphs with large spectral gap, the so-called $(n, d, \lambda)$-graphs. These graphs resemble a random graph since they are good expanders and have quite uniform

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edge distribution. For an extensive compilation of results on pseudo-random graphs the reader is referred to the survey of Krivelevich and Sudakov [KS06].

The best known result for Hamiltonicity on pseudo-random graphs is due to Krivelevich and Sudakov [KS03], where it was proved that for $d$ as low as polylog$(n)$, with sufficiently small $\lambda$, $(n,d,\lambda)$-graphs are Hamiltonian. Although those graphs are very sparse, the results for binomial random graphs and random regular graphs are evidence that even sparser pseudo-random graphs should be Hamiltonian.

**Conjecture 1** ([KS03])**.** *There exists a constant $C>0$ such that for large enough $n$ any $(n,d,\lambda)$-graph with $d/\lambda > C$ is Hamiltonian.*

We remark that there are graphs of constant degree $d$ for which $\lambda = \Theta(\sqrt{d})$. In particular, if the above conjecture is true, graphs with (large) constant degree and large spectral gap would be Hamiltonian.

In this note we consider a kind of relaxation of the Hamiltonicity problem. Instead of trying to prove the existence of a spanning cycle or path, we look for spanning trees having low degree. A recent related result of Alon, Krivelevich and Sudakov [AKS07] concerns *almost spanning* trees of bounded degree. In [AKS07] it is shown that for every $\varepsilon > 0$ and $\Delta \geq 2$, $(n,d,\lambda)$-graphs with sufficiently large $d$ and sufficiently small $\lambda$ contain all trees of size $(1-\varepsilon)n$ and degree at most $\Delta$. In contrast we show the existence of a spanning tree of degree at most 3 in $(n,d,\lambda)$ with sufficiently small $\lambda$.

Our proof is based on Pósa's rotation technique, which is frequently used on Hamiltonicity problems. We first establish a minimality criteria for spanning trees and then show that taking a minimal spanning tree under these criteria in a pseudo-random graph implies that the tree has degree at most 3.

## 2. Preliminaries

The only properties of pseudo-random graphs we shall be concerned with are the edge distribution and the vertex expansion. Namely, our result holds for graphs where the number of edges crossing large disjoint sets is proportional to the density of the graph and, moreover, every small set has a neighborhood of much larger size.

**Property 2.** An $n$-vertex graph $G=(V,E)$ satisfies *property* $\mathcal{P}(n,\alpha,d)$ if $\min_{S\subseteq V\,:\,|S|\leq \alpha n} |\Gamma(S)|/|S| \geq 10$ and, for all $X, Y \subseteq V$, with $|X|,|Y| \geq \alpha n$, we have

$$\left| e(X,Y) - \frac{d}{n}|X|\,|Y| \right| \leq \frac{\alpha d}{2}\sqrt{|X|\,|Y|}.$$

**Definition 3.** A graph is called *good* if it satisfies $\mathcal{P}(n,\alpha,d)$ for some constant $\alpha \in (0, 1/10)$.

We note that in a good graph $G=(V,E)$, for small sized sets, we have guaranteed expansion, that is, given $U \subset V$, with $|U| \leq \alpha n$, we have $|\Gamma(U)| \geq 10\,|U|$. Moreover, any set $U$ having $\alpha n$ vertices must satisfy $|\Gamma(U)| \geq (1-\alpha)n$. Indeed, since $e(U, V\setminus \Gamma(U)) = 0$, if we take $Y \subseteq$

$V \setminus \Gamma(U)$ with $\alpha n$ elements, we then have

$$\left| e(U,Y) - \frac{d}{n}|U|\,|Y| \right| = d\alpha^2 n = d\alpha\sqrt{|U|\,|Y|},$$

a contradiction. The following fact can be easily derived from this.

**Fact 4.** *Let $G$ be a good graph and suppose that $U \subseteq V(G)$ is such that $|\Gamma(U)| < 10\,|U|$. Then $|\Gamma(U)| \geq 0.9n$.*

It is a very useful fact that a large spectral gap implies edge uniformity [AS00, Corollary 9.2.5]. In any $(n,d,\lambda)$-graph with vertex set $V$, for all $U, W \subset V$ we have

$$\left| e(U,W) - d\frac{|U|\,|W|}{n} \right| \leq \lambda\sqrt{|U|\,|W|}. \tag{1}$$

The following theorem of Tanner [Tan84] relates spectral gap and vertex expansion.

**Theorem 5.** *For any $\alpha \in (0,1)$ an $(n,d,\lambda)$-graph $G = (V,E)$ satisfies*

$$\min_{\substack{S \subset V \\ |S| \leq \alpha n}} \frac{|\Gamma(S)|}{|S|} \geq \frac{d^2}{\alpha(1-\lambda^2)+\lambda^2}. \tag{2}$$

From equations (1) and (2) we conclude that any $(n,d,\lambda)$-graph satisfying $\lambda \leq d/20$ has property $\mathcal{P}(n, 1/10, d)$. There are many examples of explicit graphs where $\lambda \leq d/20$. In fact, there are many known constructions of *Ramanujan graphs*, that is, graphs for which the spectral gap is almost optimal (one always has $\lambda \geq 2\sqrt{d-1}$). In particular, all Ramanujan graphs with large enough degree are good graphs and thus satisfy the conditions of our main result.

## 3. Main Result

**Theorem 6.** *If $G$ is a good graph then there exists a spanning tree $T \subseteq G$ with $\Delta(T) \leq 3$.*

**Theorem 7** (Main result - restated)**.** *If $G$ is an $(n,d,\lambda)$-graph with $\lambda \leq d/20$ then there exists a spanning tree $T \subseteq G$ with $\Delta(T) \leq 3$.*

To prove Theorem 6 we shall generalize to trees the concept of *path rotations* developed by Pósa [Pós76].

**Definition 8** (Rotations)**.** Let $T \subset G$ be a tree. The set of *leaves* (vertices of degree 1 in $T$) is denoted by $l(T)$. Suppose that $u \in l(T)$ and $v \in V(T)$ are such that $e = \{u,v\} \in E(G) \setminus E(T)$. Let $T' = T - f + e \subset G$ with $e \neq f \in E(T)$ being the edge incident to $v$ in the unique cycle of $T + e$. Such a tree $T'$ is called an *elementary rotation* (ER) of $T$. We say that the edge $f$ is the *broken edge* and $v$ is the *pivot* of the rotation.

A general *rotation* is a tree obtained by removing an arbitrary edge of the unique cycle in $T + e$.

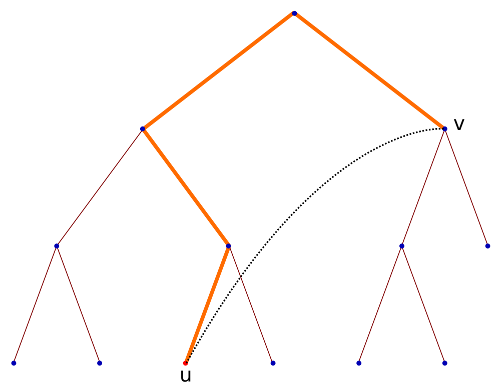


FIGURE 1. By adding an edge $\{u, v\}$ to the tree a cycle is formed with the edges in bold. Removing the bold edge incident to $v$ we obtain a new tree which is an elementary rotation of the original tree. If instead we remove an arbitrary bold edge, we obtain a general rotation of the original tree.

**Definition 9.** A tree $T \subseteq G$ is called *$\ell$-minimal* if, among all spanning trees in $G[V(T)]$, it has minimum number of leaves.

Notice that an elementary rotation preserves the vertex set of the tree and does not increase the number of leaves. Suppose that $T$ is $\ell$-minimal. In such a case, all elementary rotations are reversible. Indeed, suppose that $T' = T - f + e$ is an elementary rotation of $T$ where $u \in l(T)$ and $e = uv$. If $f = xv$ then $x$ must be a leaf in $T'$ otherwise we contradict $\ell$-minimality. Hence, $T = T' - e + f$ is an elementary rotation of $T'$.

**Definition 10** (Similarity relation)**.** Let $W \subseteq V(G)$ be such that $G' = G[W]$ is connected. Suppose that $T$ is an $\ell$-minimal spanning tree of $G'$. We write $T \sim T'$ if $T'$ can be obtained from $T$ by a sequence of elementary rotations.

Note that the above indeed defines a relation over the family of all $\ell$-minimal spanning trees of $G'$.

**Definition 11** (Leaf closure)**.** Let $T \subset G$ be an $\ell$-minimal tree. The *leaf closure* of $T$, denoted by $\bar{l}(T)$ is given by $\bigcup_{T' \sim T} l(T')$.

Our next lemmas and facts are inspired by Pósa's Rotation-Extension technique [Pós76].

**Lemma 12.** *Given an $\ell$-minimal tree $T \subset G$, let $N = \Gamma_T(\bar{l}(T))$. Then there are no edges in $G$ between $V(T) \setminus (\bar{l}(T) \cup N)$ and $\bar{l}(T)$.*

*Proof.* Let $v \in V(T) \setminus (\bar{l}(T) \cup N)$. Suppose that there is an edge $uv \in E(G)$ with $u \in \bar{l}(T)$. Let $T' \sim T$ be such that $u \in l(T')$. The set of edges incident to $v$ in $T'$ is precisely the same as in $T$ since, otherwise, a rotation that involved an edge incident to $v$ would take place in the sequence of elementary rotations from $T$ to $T'$. But if an edge (of $G$) incident to $v$ is added or removed in an ER, either $v$ or one of its neighbors in $T$ is contained in $\bar{l}(T)$, a contradiction. Note that, in particular, we have $\Gamma_{T'}(v) = \Gamma_T(v)$.

Since $e = uv \in E(G) \setminus E(T')$ (as otherwise $u \in \Gamma_{T'}(v) = \Gamma_T(v)$ and thus $v \in N$), we could perform an elementary rotation by adding $e$ to $T'$ and removing some edge $f \in E(T')$ incident to $v$, but this is a contradiction since some vertex of $\Gamma_{T'}(v) = \Gamma_T(v)$ would become a leaf and would belong to $\bar{l}(T)$. □

Note that in an $\ell$-minimal tree $T$ every vertex $u \in \bar{l}(T)$ satisfies $d_T(u) \leq 2$. In fact, if $d_T(u) \geq 3$ then $d_{T'}(u) = d_T(u)$ for all $T' \sim T$. This follows because when an edge incident to $u$ is broken, $u$ must be the pivot of the rotation. If that is not the case, we obtain a contradiction to the $\ell$-minimality of $T$ by reducing the number of leaves through a sequence of elementary rotations. In fact, every elementary rotation changes the degree of one leaf to degree 2 and turns a vertex of degree 2 into one leaf (preserving all other degrees).

From the simple observations above we get the following facts.

**Fact 13.** *If $T \subset G$ is an $\ell$-minimal tree, we have $|N| \leq 2\,|\bar{l}(T)|$.*

**Fact 14.** *If $T \subset G$ is an $\ell$-minimal tree then every $T' \sim T$ has the same degree sequence as $T$. Moreover, if $v \in V(T)$ has degree $d_T(v) \geq 3$ then $d_{T'}(v) = d_T(v)$ for all $T' \sim T$.*

The following technical lemma shows that, by taking a tree that is minimum with respect to certain criteria, it is possible to find a maximum degree vertex for which its subtrees have roughly balanced sizes. For this purpose, define $\gamma(v) = \max\{|T_v| \,:\, T_v \text{ is a tree in } T - v\}$.

**Lemma 15.** *Let $G$ be a good graph and $T \subset G$ be a spanning tree that is minimal according to the following criteria (in order of importance)*

(1) *number of leaves;*
(2) $\sum_{v\in V} d_T(v)^2$;
(3) $\omega(T) = \min_{v\in V\,:\,d_T(v)=\Delta(T)} \gamma(v)$.

*If $\Delta(T) \geq 4$ then $\omega(T) \leq 2n/3$.*

Once the above lemma is established, we can decompose $T$ into two edge-disjoint subtrees $T_0$, $T_1$ of balanced size for which $V(T_0) \cap V(T_1) = \{v\}$ for some $v$ having maximum degree in $T$. Later we shall prove that unless $\Delta(T) \leq 3$, such a decomposition implies a violation of the minimality of $T$.

*Proof of Lemma 15.* Suppose, for the sake of a contradiction, that $\Delta(T) \geq 4$ and $T$ is minimal with respect to criteria (1)-(3) and $\omega(T) > 2n/3$. Let $v \in V$ be a vertex of maximum degree in $T$ such that $T_v$, the largest tree in $T - v$, has size $\omega(T)$.

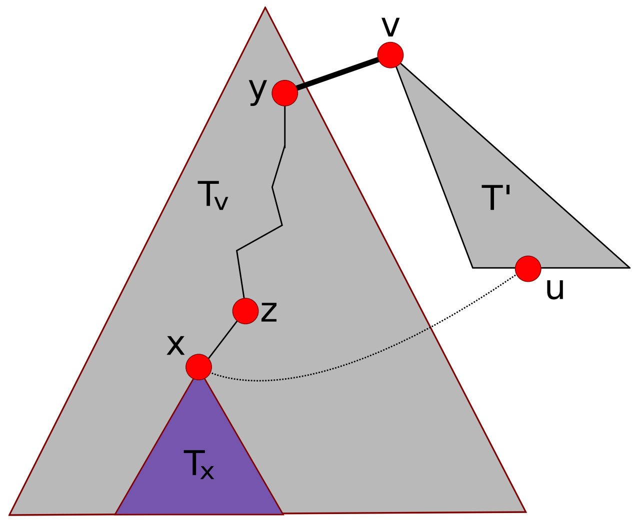


FIGURE 2. A tree $T$ that is minimal with respect to (1)-(3). The vertex $v$ is a vertex with $d_T(v) = \Delta(T)$ such that $\omega(T) = \gamma(v)$; the other marked vertices are all referred in the Proof of Lemma 15.

Let $W = V \setminus V(T_v)$, where $V = V(G)$. Observe that $T[W]$ is $\ell$-minimal since $v$ is not a leaf of $T[W]$: in particular, every leaf of $T[W]$ is also a leaf of $T$. Therefore we may apply Lemma 12 to $T[W]$.

Let $U$ denote the leaf closure of $T[W]$ and $N = \Gamma_{T[W]}(U)$. From Lemma 12, we get that there is no $U$–$W \setminus (N \cup U)$ edge. Suppose that there is no edge from $U$ to $V(T_v)$ (hence no $U$–$V \setminus (N \cup U)$ exists). Since $\Gamma(U) \subseteq U \cup N$, Fact 13 implies $|\Gamma(U)| \leq 3\,|U|$. From Fact 4, it follows that $|\Gamma(U)| \geq 0.9n$, which contradicts $\Gamma(U) \subseteq W$ with $|W| \leq n/3$. Hence, there must be a vertex in $U$ sending an edge to $T_v$.

Let $T' \sim T[W]$ be a tree having a leaf $u$ that sends an edge to $T_v$. We note that by replacing $T[W]$ with $T'$ in the tree $T$ affects neither the first nor the second minimality criteria, since elementary rotations of minimal trees preserve the degree sequence (Fact 14). As for the third criterion, it is enough to argue that the degree of $v$ in $T'$ is the same as in $T[W]$. Since $d_{T[W]}(v) \geq 3$ (recall $d_T(v) = \Delta(T) \geq 4$), this follows from Fact 14.

For simplicity, we shall assume from now on that $T[W] = T'$. Let $x \in V(T_v)$ be a neighbor of $u$. If $d_T(x) \leq \Delta - 2$, one can obtain $T^*$ with the same number of leaves and reduce the second criterion by performing a general rotation: add $e = ux$ and remove $f$ from $T$, where $f = vy$ is the edge connecting $v$ to $T_v$. Note that $d_T(y) = 2$ since, otherwise, $T^*$ would have fewer leaves than $T$. The sum of squares of the degrees would decrease by

$$\begin{aligned} D &= d_T(v)^2 - d_{T*}(v)^2 + d_T(x)^2 - d_{T*}(x)^2 \\ &+ d_T(u)^2 - d_{T*}(u)^2 + d_T(y)^2 - d_{T*}(y)^2 \\ &= 2\Delta - 2d_{T^*}(x) \\ &\geq 2\Delta - 2(\Delta - 1) = 2. \end{aligned} \tag{3}$$

It follows that $d_T(x) \geq \Delta - 1$. Suppose that $xz \in E(T)$ is in the path from $x$ to $v$ in $T$. Consider the subtree $T_x \subset T$ containing $x$ in the forest $T - xz$. If $|V(T_x)| \leq n/3$, then the elementary rotation $T^* = T - xz + ux$ preserves the degree sequence and decreases the third criterion, since $d_{T^*}(v) = \Delta$ and the largest tree in $T^* - v$ is strictly smaller than $T_v$ (as $T_x$ is a sub-tree of $T_v$ which is now joined to another subtree with less than $n/3$ vertices).

On the other hand, if $|V(T_x)| > n/3$ then we would have $d_T(x) = \Delta - 1$ as otherwise the largest tree in $T - x$ would be smaller than $T_v$, which would contradict the assumption on $v$. But in this case, performing a general rotation, we would obtain $T^* = T - vy + ux$. The degree sequence is preserved by this rotation and $x$ would have maximum degree in $T^*$. Since the largest tree in $T^* - x$ would then be smaller than $T_v$, we again have a contradiction. It follows that $\omega(T) \leq 2n/3$. □

We now use Lemma 15 to obtain a tree $T$ that is minimal with respect to criteria (1)-(3) which is the edge disjoint union of trees $T_0$ and $T_1$ having somewhat balanced sizes. Let $T'_1, T'_2, \ldots, T'_\Delta$ be the collection of trees in $T - v$, where $v$ is a vertex of maximum degree $\Delta$ such that $|T'_i| \leq 2n/3$ for all $i$. Let $S(J) = \sum_{j \in J} |T'_j|$. Take $I \subset [\Delta]$ to be such that

$$D = \big|S(I) - S([\Delta] \setminus I)\big|$$

is minimum. Without loss of generality, assume that $S(I) \geq S([\Delta] \setminus I)$. For every $i \in I$, if we remove $i$ from $I$, we cannot decrease the above difference, meaning $D \leq |T'_i|$. In particular, $S([\Delta] \setminus I) \geq S(I) - |T'_i|$. If $I$ contains only one element, we know $S(I) \leq 2n/3$. If $I$ is not a singleton, then there exists $i$ such that $|T'_i| \leq S(I)/2$. In both cases it follows that $n - S(I) = 1 + S([\Delta] \setminus I) \geq S(I) - D \geq S(I) - |T'_i| \geq S(I)/2$.

If $I$ is a singleton, take $j \in [\Delta] \setminus I$ with $|T'_j|$ minimum and add it to $I$. Notice that this increases $S(I)$ by at most $S([\Delta] \setminus I)/(\Delta - 1) \leq [n - S(I)]/3$. If $|I| = \Delta - 1$ then take $j \in I$ with $|T'_j|$ minimum and remove it from $I$. In both cases, $\max\{S(I), S([\Delta] \setminus I)\} \leq 7n/9$.

We set $T_0$ to be the union of all trees $T'_i$, $i \in I$, together with the edges connecting those trees to $v$. Let $T_1$ be the tree with edges $E(T) - E(T_0)$.

*Proof of Theorem 6.* Suppose, for the sake of a contradiction, that a spanning tree $T$ that is minimal with respect to criteria (1)-(3) has maximum degree $\Delta(T) \geq 4$. Consider the above decomposition of $T$ into two edge-disjoint trees $T_0$ and $T_1$.

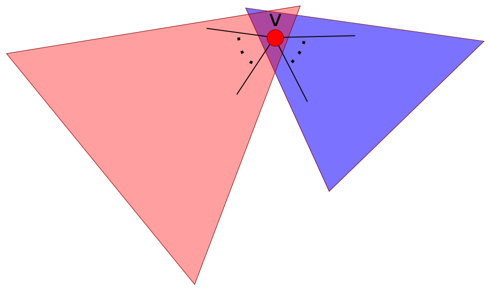


Figure 3. Representation of the decomposition of $T$ into two balanced edge-disjoint trees. Notice that $v$ is the only vertex shared by $T_0$ and $T_1$ and that $v$ has degree at least 2 in both.

Let $L = |l(T)|$ and $L_i = |V(T_i) \cap l(T)|$, for $i = 0, 1$. Note that the vertex set of the trees $T_0$ and $T_1$ intersect precisely at $v$ (which is not in $l(T)$) and therefore $L = L_0 + L_1$. Let $j$ be such that $L_j \geq L_{1-j}$. Denote by $R$ the number of elements $w \in V(T_{1-j}) - v$ such that $d_T(w) \geq \Delta - 1 \geq 3$. By adding up the degrees in the tree $T_{1-j}$, we get

$$\begin{aligned} 2(|V(T_{1-j})| - 1) = 2\,|E(T_{1-j})| &= d_{T_{1-j}}(v) + \sum_{w \in V(T_{1-j}) \setminus \{v\}} d_T(w) \\ &\geq L_{1-j} \cdot 1 + R \cdot (\Delta - 1) + (|V(T_{1-j})| - 1 - L_{1-j} - R) \cdot 2 \\ &= 2(|V(T_{1-j})| - 1) + R(\Delta - 3) - L_{1-j}. \end{aligned}$$

It follows that $R \leq L_{1-j}/(\Delta - 3)$.

Let $U'$ be the set of leaves that one can obtain by performing elementary rotations which keep $T_{1-j}$ fixed, namely

$$U' = \bigcup_{T' \sim T \,:\, T_{1-j} \subset T'} l(T').$$

Set $U = U' \setminus l(T_{1-j})$. As seen in the proof of Lemma 15, any neighbor of $U$ in $V(T_{1-j}) - v$ must have degree at least $\Delta - 1$ in $T_{1-j}$, as otherwise we would contradict the minimality of the sum of squares of the degrees in $T$. Hence, there are at most $R \leq L_{1-j}/(\Delta - 3) \leq L_j/(\Delta - 3)$ such neighbors.

Let $N = \Gamma_T(U)$. Observe that, by construction, $T_j$ does not have $v$ as leaf. In particular $T_v$ must be an $\ell$-minimal tree since any tree on $V(T_v)$ with less leaves than $T_v$ could be extended to a spanning tree of $G$ with less leaves than $T$. By Lemma 12 there are no $U$–$V(T_j) \setminus (N \cup U)$ edges in $G$. It follows that the number of neighbors of $U \supseteq l(T) \cap V(T_j)$ in $G$ is at most $R + |N \cup U| \leq L_j + 3\,|U| \leq 4\,|U|$. But by Fact 4, we must have $|\Gamma(U)| \geq 0.9n$ and, moreover, at least $0.9n - R$ elements of $\Gamma(U)$ are contained in $T_j$.

Let $n_i = |V(T_i)|$, $i = 0, 1$. Then $n_0 + n_1 = n + 1$ and $R + L_{1-j} \leq n_{1-j} - 1$. Using that $R \leq L_{1-j}$, we get that

$$n_j \geq 0.9n - R \geq 0.9(n_0 + n_1 - 1) - 0.5(n_{1-j} - 1) = 0.9n_j + 0.4n_{1-j} - 0.4.$$

But this implies that $n_j \geq 4n_{1-j} - 4 = 4(n - n_j)$ and thus $n_j \geq 4n/5$, a contradiction since by construction $\max\{n_0, n_1\} \leq 7n/9 + 1$. □

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Emory University – Department of Mathematics and Computer Science, 400 Dowman Dr., W401, Atlanta, GA 30322

*E-mail address*: ddellam@mathcs.emory.edu