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A Dimensional Split Preconditioner for the Stokes and Navier-Stokes Equations

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A DIMENSIONAL SPLIT PRECONDITIONER FOR STOKES AND LINEARIZED NAVIER–STOKES EQUATIONS

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Abstract. In this paper we introduce a new preconditioner for linear systems of saddle point type arising from the numerical solution of the Navier–Stokes equations. Our approach is based on a dimensional splitting of the problem along the components of the velocity field, resulting in a convergent fixed-point iteration. The basic iteration is accelerated by a Krylov subspace method like restarted GMRES. The corresponding preconditioner requires at each iteration the solution of a set of discrete scalar elliptic equations, one for each component of the velocity field. Numerical experiments showing mesh-independent convergence on a standard finite element discretization of the Stokes and Oseen problems are included.

Key words. saddle point problems, matrix splittings, iterative methods, preconditioning, Stokes problem, Oseen problem

AMS subject classifications. Primary 65F10, 65N22, 65F50. Secondary 76M.

1. Introduction.

We consider the solution of the incompressible Navier–Stokes equations governing the flow of viscous Newtonian fluids. For an open bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) with boundary $\partial\Omega$, time interval $[0, T]$, and data $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{u}_0$, the goal is to find a velocity field $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ and pressure field $p = p(\mathbf{x}, t)$ such that

$$\frac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{on } \Omega \times (0, T], \tag{1.1}$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{on } \Omega \times [0, T], \tag{1.2}$$

$$\mathbf{u} = \mathbf{g} \quad \text{on } \partial\Omega \times [0, T], \tag{1.3}$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}) \quad \text{on } \Omega, \tag{1.4}$$

where ν is the kinematic viscosity, Δ is the vector Laplacian, ∇ is the gradient and div the divergence. Implicit time discretization and linearization of the Navier–Stokes system by Picard fixed-point iteration result in a sequence of (generalized) Oseen problems of the form

$$\sigma\mathbf{u} - \nu\Delta\mathbf{u} + (\mathbf{v}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \tag{1.5}$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \tag{1.6}$$

$$\mathbf{u} = \mathbf{g} \quad \text{on } \partial\Omega, \tag{1.7}$$

where $\mathbf{v}$ is a known velocity field from a previous iteration or time step (the 'wind') and σ is proportional to the reciprocal of the time step ($\sigma = 0$ for a steady problem). When $\mathbf{v} = \mathbf{0}$ we have a (generalized) Stokes problem.

Spatial discretization of the Stokes or Oseen problem using LBB-stable finite elements (cf. [17, 18]) results in large, sparse linear systems in *saddle point form*:

$$\begin{bmatrix} A & B^T \\ -B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ -g \end{bmatrix}, \quad \text{or} \quad \mathcal{A}\mathbf{x} = \mathbf{b}, \tag{1.8}$$

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where now $\mathbf{u}$ and p represent the discrete velocity and pressure, respectively, A is the discretization of the diffusion, convection, and time-dependent terms, B^T is the discrete gradient, B the (negative) discrete divergence, and $\mathbf{f}$ and g contain forcing and boundary terms.

The efficient solution of (1.8) calls for rapidly convergent iterative methods. Much work has been done in developing efficient preconditioners for Krylov subspace methods applied to this problem; see, e.g., [4, 5, 7, 13, 14, 16, 17, 21]. The ultimate goal is to develop robust solvers with optimal complexity. In particular, the rate of convergence should be independent of the mesh size h. For the Oseen problem, the rate of convergence should also depend only weakly on the kinematic viscosity ν (equivalently, of the Reynolds number $Re = O(\nu^{-1})$), although this goal is difficult to achieve in practice.

2. Dimensional splitting. For simplicity, in this paper we limit ourselves to the 2D case. Extension to the 3D case is possible (see section 7), but will not be described here. The system matrix $\mathcal{A}$ admits the following splitting:

$$\mathcal{A} = \begin{bmatrix} A_1 & 0 & B_1^T \\ 0 & A_2 & B_2^T \\ -B_1 & -B_2 & 0 \end{bmatrix} = \begin{bmatrix} A_1 & 0 & B_1^T \\ 0 & 0 & 0 \\ -B_1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_2 & B_2^T \\ 0 & -B_2 & 0 \end{bmatrix} = \mathcal{A}_1 + \mathcal{A}_2 \,. \tag{2.1}$$

Here each diagonal submatrix A_i is a scalar discrete convection-diffusion-reaction operator:

$$A_i = \sigma M + \nu L + N_i \qquad (i = 1, 2), \tag{2.2}$$

and B_1^T, B_2^T are discretizations of the partial derivatives $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, respectively. Note that A_1 and A_2 act, respectively, on u (the x-component of the velocity field $\mathbf{u}$) and on v (the y-component of $\mathbf{u}$). Denoting by n_1, n_2 and m the number of degrees of freedom of u, v and p, respectively, then $A_1 \in \mathbb{R}^{n_1 \times n_1}$, $A_2 \in \mathbb{R}^{n_2 \times n_2}$, $B_1 \in \mathbb{R}^{m \times n_1}$ and $B_2 \in \mathbb{R}^{m \times n_2}$. Thus, $\mathcal{A} \in \mathbb{R}^{(n+m)\times(n+m)}$ with $n = n_1 + n_2$. In (2.2), M denotes the velocity mass matrix, L the discrete (negative) Laplacian, and N_i the convective terms. Note that for the discrete Stokes problem the convective terms are absent ($N_i = 0$) so that each A_i is symmetric and positive definite. For the discrete Oseen problem $A_i \neq A_i^T$, but $A_i + A_i^T$ is positive definite ($i = 1, 2$). As a consequence, $\mathcal{A}_1$ and $\mathcal{A}_2$ in (2.1) are nonsymmetric but positive semidefinite, in the sense that $\mathcal{A}_1 + \mathcal{A}_1^T$ and $\mathcal{A}_2 + \mathcal{A}_2^T$ are both symmetric positive semidefinite. In particular, $\mathcal{A}_1$ and $\mathcal{A}_2$ are singular. We refer to (2.1) as to a *dimensional splitting*, since $\mathcal{A}_1$ contains terms that correspond to the x-component of the solution and $\mathcal{A}_2$ contains terms that correspond to the y-component of the solution. Although this splitting is somewhat reminiscent of ADI (alternating direction implicit) methods [28], it is actually quite different since we do not split the operators A_i into their constituent components. To distinguish it from ADI splitting, we refer to (2.1) as to *dimensional splitting* (or DS for short). We further mention that our approach is also different from previous ADI schemes for saddle point problems, such as those described in [10] and [12].

Let now $\alpha > 0$ be a parameter, and denote by $\mathcal{I}$ the identity matrix of order $n_1 + n_2 + m$. Then $\mathcal{A}_1 + \alpha\mathcal{I}$ and $\mathcal{A}_2 + \alpha\mathcal{I}$ are both nonsingular, nonsymmetric, and positive definite. Consider the two splittings of $\mathcal{A}$,

$$\mathcal{A} = (\mathcal{A}_1 + \alpha\mathcal{I}) - (\alpha\mathcal{I} - \mathcal{A}_2) \qquad \text{and} \qquad \mathcal{A} = (\mathcal{A}_2 + \alpha\mathcal{I}) - (\alpha\mathcal{I} - \mathcal{A}_1) \,.$$

Associated to these splittings is the alternating iteration

$$(\mathcal{A}_1 + \alpha\mathcal{I})\mathbf{x}^{k+\frac{1}{2}} = (\alpha\mathcal{I} - \mathcal{A}_2)\mathbf{x}^k + \mathbf{b}, \tag{2.3}$$

$$(\mathcal{A}_2 + \alpha\mathcal{I})\mathbf{x}^{k+1} = (\alpha\mathcal{I} - \mathcal{A}_1)\mathbf{x}^{k+\frac{1}{2}} + \mathbf{b}, \tag{2.4}$$

$(k = 0, 1, \ldots)$. Eliminating $\mathbf{x}^{k+\frac{1}{2}}$ from these, we can rewrite (2.3)-(2.4) as the stationary scheme

$$\mathbf{x}^{k+1} = \mathcal{T}_\alpha \mathbf{x}^k + \mathbf{c}, \qquad k = 0, 1, \ldots$$

where $\mathcal{T}_\alpha$ is the iteration matrix

$$\mathcal{T}_\alpha = (\mathcal{A}_2 + \alpha\mathcal{I})^{-1}(\alpha\mathcal{I} - \mathcal{A}_1)(\mathcal{A}_1 + \alpha\mathcal{I})^{-1}(\alpha\mathcal{I} - \mathcal{A}_2) \tag{2.5}$$

and $\mathbf{c}$ is a certain vector. As in [8], one can show that there is a unique splitting $\mathcal{A} = \mathcal{P}_\alpha - \mathcal{Q}_\alpha$ with $\mathcal{P}_\alpha$ nonsingular such that the iteration matrix $\mathcal{T}_\alpha$ is the matrix induced by that splitting, i.e., $\mathcal{T}_\alpha = \mathcal{P}_\alpha^{-1}\mathcal{Q}_\alpha = \mathcal{I} - \mathcal{P}_\alpha^{-1}\mathcal{A}$. Furthermore, $\mathbf{c} = \mathcal{P}_\alpha^{-1}\mathbf{b}$. Matrices $\mathcal{P}_\alpha$ and $\mathcal{Q}_\alpha$ are given by

$$\mathcal{P}_\alpha = \tfrac{1}{2\alpha}(\mathcal{A}_1 + \alpha\mathcal{I})(\mathcal{A}_2 + \alpha\mathcal{I})\,, \qquad \mathcal{Q}_\alpha = \tfrac{1}{2\alpha}(\alpha\mathcal{I} - \mathcal{A}_1)(\alpha\mathcal{I} - \mathcal{A}_2). \tag{2.6}$$

We refer to iteration (2.3)-(2.4) as the *DS iteration*, and to $\mathcal{P}_\alpha$ as the *DS preconditioner*. Besides the already mentioned resemblance to the Peaceman–Rachford ADI method, the DS iteration bears some resemblance to another alternating method, the Hermitian and skew-Hermitian splitting (HSS) iteration [2, 3, 4, 6]. While the HSS method has proved quite successful in solving such problems as the generalized Stokes problem and the rotation form of the Navier–Stokes equations (see [6]), it is not well-suited for the standard (convection) form (1.1)-(1.2). This limitation of HSS was one of the main motivations for introducing the DS approach.

3. Convergence of the fixed-point iteration. We now prove that, under standard assumptions on the saddle point problem (1.8), the alternating iteration (2.3)-(2.4) converges to the solution of (1.8) for any choice of $\alpha > 0$ and for alll initial guesses. First we state two auxiliary results. The first one is classical, and is known as *Kellogg's Lemma*; see [20]. In the following, a (not necessarily Hermitian) matrix $A \in \mathbb{C}^{n\times n}$ is said to be *positive definite (semidefinite)* if the Hermitian matrix $A + A^*$ is positive definite (resp., semidefinite) in the usual sense.

LEMMA 3.1. *Let $A \in \mathbb{C}^{n\times n}$ be positive semidefinite. Then*

$$\|(\alpha I_n + A)^{-1}(\alpha I_n - A)\|_2 \leq 1$$

for all $\alpha > 0$. Furthermore, if A is positive definite then

$$\|(\alpha I_n + A)^{-1}(\alpha I_n - A)\|_2 < 1$$

for all $\alpha > 0$.

LEMMA 3.2. *Assume that the (1,1) block A in (1.8) has positive definite symmetric part and that B has full row rank. Then the following are equivalent:*

(i) The matrix

$$\mathcal{C}_\alpha := \begin{bmatrix} A_1 & \frac{1}{\alpha^2}B_1^T B_2 A_2 & B_1^T + \frac{1}{\alpha^2}B_1^T B_2 B_2^T \\ 0 & A_2 & B_2^T \\ -B_1 & -B_2 & 0 \end{bmatrix} \tag{3.1}$$

has no purely imaginary eigenvalues.

(ii) The spectral radius of the iteration matrix $\mathcal{T}_\alpha$ in (2.5) is strictly less than unity.

Proof. First we note that under the assumptions made on A and B, the matrix $\mathcal{A}$ in (1.8) is nonsingular; see [4, Lemma 1.1]. Let λ be an eigenvalue of the iteration matrix $\mathcal{T}_\alpha = \mathcal{I} - \mathcal{P}_\alpha^{-1}\mathcal{A}$. Then $\lambda = 1 - \mu$ where μ is a generalized eigenvalue of the matrix pencil $(\mathcal{A}, \mathcal{P}_\alpha)$; that is, there exists a vector $\mathbf{x} \neq \mathbf{0}$ such that $\mathcal{A}\mathbf{x} = \mu\mathcal{P}_\alpha\mathbf{x}$. Expanding the right-hand side, we get

$$\mathcal{A}\mathbf{x} = \frac{\mu}{2\alpha}\left(\mathcal{A}_1\mathcal{A}_2 + \alpha\mathcal{A} + \alpha^2\mathcal{I}\right)\mathbf{x}.$$

Collecting terms in $\mathcal{A}$, we rewrite this as

$$\left(1 - \frac{1}{2}\mu\right)\mathcal{A}\mathbf{x} = \frac{\mu\alpha}{2}\left(\mathcal{I} + \frac{1}{\alpha^2}\mathcal{A}_1\mathcal{A}_2\right)\mathbf{x}. \tag{3.2}$$

Since both $\mathcal{A}$ and $\mathcal{P}_\alpha$ are nonsingular, it must be $\mu \neq 0$. Also, it must be $1 - \frac{1}{2}\mu \neq 0$ for otherwise (3.2) implies that $\left(\mathcal{I} + \frac{1}{\alpha^2}\mathcal{A}_1\mathcal{A}_2\right)\mathbf{x} = \mathbf{0}$ has a nonzero solution, but this is impossible since

$$\mathcal{G} := \mathcal{I} + \frac{1}{\alpha^2}\mathcal{A}_1\mathcal{A}_2 = \begin{bmatrix} I_{n_1} & -\frac{1}{\alpha^2}B_1^TB_2 & 0 \\ 0 & I_{n_2} & 0 \\ 0 & 0 & I_m \end{bmatrix}$$

is clearly nonsingular. Hence, $\mu \neq 2$ and we can set (as in [24, p. 378])

$$\theta := \frac{\mu\alpha}{2-\mu}, \quad \text{from which} \quad \mu = 2 - \frac{2\alpha}{\theta+\alpha} = \frac{2\theta}{\theta+\alpha}.$$

Hence, the generalized eigenproblem (3.2) can be reformulated as

$$\mathcal{A}\mathbf{x} = \theta\mathcal{G}\mathbf{x},$$

that is,

$$\begin{bmatrix} A_1 & 0 & B_1^T \\ 0 & A_2 & B_2^T \\ -B_1 & -B_2 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \\ p \end{bmatrix} = \theta\begin{bmatrix} I_{n_1} & -\frac{1}{\alpha^2}B_1^TB_2 & 0 \\ 0 & I_{n_2} & 0 \\ 0 & 0 & I_m \end{bmatrix}\begin{bmatrix} u \\ v \\ p \end{bmatrix},$$

or

$$\mathcal{G}^{-1}\mathcal{A}\mathbf{x} = \theta\mathbf{x}, \quad \text{where} \quad \mathcal{G}^{-1} = \begin{bmatrix} I_{n_1} & \frac{1}{\alpha^2}B_1^TB_2 & 0 \\ 0 & I_{n_2} & 0 \\ 0 & 0 & I_m \end{bmatrix}.$$

This eigenproblem is precisely $\mathcal{C}_\alpha\mathbf{x} = \theta\mathbf{x}$, where $\mathcal{C}_\alpha = \mathcal{G}^{-1}\mathcal{A}$ is the matrix in (3.1). Note that $\mathcal{C}_\alpha$ is necessarily nonsingular. Recall now that the eigenvalues of $\mathcal{P}_\alpha^{-1}\mathcal{A}$ are of the form $\mu = 2\theta/(\theta+\alpha)$. It must be $|1-\mu| \leq 1$, since $\lambda = 1 - \mu$ is an eigenvalue of the iteration matrix

$$\mathcal{T}_\alpha = \mathcal{I} - \mathcal{P}_\alpha^{-1}\mathcal{A} = (\alpha\mathcal{I} + \mathcal{A}_2)^{-1}(\alpha\mathcal{I} - \mathcal{A}_1)(\alpha\mathcal{I} + \mathcal{A}_1)^{-1}(\alpha\mathcal{I} - \mathcal{A}_2),$$

which is similar to

$$\widehat{\mathcal{T}}_\alpha = (\alpha\mathcal{I} - \mathcal{A}_1)(\alpha\mathcal{I} + \mathcal{A}_1)^{-1}(\alpha\mathcal{I} - \mathcal{A}_2)(\alpha\mathcal{I} + \mathcal{A}_2)^{-1},$$

hence $\varrho(\mathcal{T}_\alpha)$, the spectral radius of $\mathcal{T}_\alpha$, satisfies

$$\varrho(\mathcal{T}_\alpha) = \varrho(\widehat{\mathcal{T}}_\alpha) \leq \|(\alpha\mathcal{I} - \mathcal{A}_1)(\alpha\mathcal{I} + \mathcal{A}_1)^{-1}\|_2 \|(\alpha\mathcal{I} - \mathcal{A}_2)(\alpha\mathcal{I} + \mathcal{A}_2)^{-1}\|_2 \leq 1. \quad (3.3)$$

The last inequality is an immediate consequence of Kellogg's Lemma.

Denoting by $\Re(\theta)$ and $\Im(\theta)$ the real and imaginary part of θ, respectively, we claim that $\varrho(\mathcal{T}_\alpha) < 1$ if and only if $\Re(\theta) \neq 0$ for every eigenvalue θ of $\mathcal{C}_\alpha$; equivalently, $\varrho(\mathcal{T}_\alpha) = 1$ if and only if there exists at least one θ with $\Re(\theta) = 0$. Indeed, we have

$$|1-\mu| = 1 \quad \Leftrightarrow \quad \left|\frac{2\theta}{\theta+\alpha} - 1\right| = 1 \quad \Leftrightarrow \quad \left|\frac{\theta-\alpha}{\theta+\alpha}\right| = 1 \quad \Leftrightarrow \quad |\theta-\alpha| = |\theta+\alpha|.$$

The last equality can be rewritten as $(\Re(\theta)-\alpha)^2 + \Im(\theta)^2 = (\Re(\theta)+\alpha)^2 + \Im(\theta)^2$, or

$$(\Re(\theta)-\alpha)^2 = (\Re(\theta)+\alpha)^2 \quad \Leftrightarrow \quad 4\alpha\Re(\theta) = 0 \quad \Leftrightarrow \quad \Re(\theta) = 0.$$

Therefore, $\varrho(\mathcal{T}_\alpha) = 1$ if and only if $\mathcal{C}_\alpha$ has at least one purely imaginary eigenvalue. The proof is complete. □

We are now in a position to prove the following convergence result.

THEOREM 3.3. *Under the assumptions of Lemma 3.2, the iteration (2.3)-(2.4) is unconditionally convergent; that is, $\varrho(\mathcal{T}_\alpha) < 1$ for all $\alpha > 0$.*

Proof. By Lemma 3.2, it suffices to show that for all $\alpha > 0$, the matrix $\mathcal{C}_\alpha$ in (3.1) has no purely imaginary eigenvalues. We will argue by contradiction. Recall that $\mathcal{C}_\alpha$ is nonsingular. Thus, let $\theta \neq 0$ be an eigenvalue of $\mathcal{C}_\alpha$ corresponding to an eigenvector $\mathbf{x} = [u; v; p]$, where $u \in \mathbb{C}^{n_1}$, $v \in \mathbb{C}^{n_2}$ and $p \in \mathbb{C}^m$ are not all equal to zero. Expanding $\mathcal{C}_\alpha\mathbf{x} = \theta\mathbf{x}$ we obtain

$$A_1u + \frac{1}{\alpha^2}B_1^TB_2A_2v + B_1^Tp + \frac{1}{\alpha^2}B_1^TB_2B_2^Tp = \theta u, \quad (3.4)$$

$$A_2v + B_2^Tp = \theta v, \quad (3.5)$$

$$-B_1u - B_2v = \theta p. \quad (3.6)$$

Assuming that the eigenvector $\mathbf{x}$ has been normalized so that $\|\mathbf{x}\|_2 = 1$, we have

$$\theta = \mathbf{x}^*\mathcal{C}_\alpha\mathbf{x}, \quad \bar{\theta} = \mathbf{x}^*\mathcal{C}_\alpha^T\mathbf{x} \quad \text{and} \quad \Re(\theta) = \frac{\theta+\bar{\theta}}{2} = \frac{1}{2}\mathbf{x}^*\left(\mathcal{C}_\alpha + \mathcal{C}_\alpha^T\right)\mathbf{x}.$$

Therefore, letting $H_1 = \left(A_1 + A_1^T\right)/2$ and $H_2 = \left(A_2 + A_2^T\right)/2$, we find after some easy algebraic manipulations

$$\Re(\theta) = u^*H_1u + v^*H_2v + \frac{1}{2\alpha^2}\left[u^*B_1^TB_2\left(A_2v + B_2^Tp\right) + \left(v^*A_2^T + p^*B_2\right)B_2^TB_1u\right].$$

First we observe that at least one between u and v must be nonzero, for otherwise (3.6), together with the fact that $\theta \neq 0$, implies $p = 0$ and thus $\mathbf{x} = \mathbf{0}$, a contradiction. Since H_1 and H_2 are symmetric positive definite we have that $u^*H_1u + v^*H_2v > 0$, therefore

$$\Re(\theta) = 0 \Rightarrow u^*B_1^TB_2\left(A_2v + B_2^Tp\right) + \left(v^*A_2^T + p^*B_2\right)B_2^TB_1u < 0, \quad (3.7)$$

showing that if θ is purely imaginary, it must necessarily be $u \neq 0$. Next, consider the case where $v = 0$. Then equations (3.4)-(3.6) reduce to

$$A_1u + B_1^Tp = \theta u, \quad (3.8)$$

$$B_2^Tp = 0, \quad (3.9)$$

$$-B_1u = \theta p, \quad (3.10)$$

that is, $\mathcal{A}\mathbf{x} = \theta\mathbf{x}$ where $\mathbf{x} = [u; 0; p] \neq \mathbf{0}$. Hence, θ is an eigenvalue of $\mathcal{A}$; but then $\Re(\theta) > 0$ by virtue of Lemma 1.1 in [4]. So it must be $v \neq 0$. Furthermore, if $p = 0$, then equation (3.5) becomes $A_2 v = \theta v$, hence $\Re(\theta) > 0$ since A_2 is positive definite (has positive definite symmetric part) by assumption, and therefore all its eigenvalues have positive real part. Thus, it must be $u \neq 0$, $v \neq 0$ and $p \neq 0$. Now, using (3.5) we rewrite the necessary condition in (3.7) in the form

$$\theta u^* B_1^T B_2 v + \bar{\theta} v^* B_2^T B_1 u = \theta (B_1 u)^* B_2 v + \bar{\theta} (B_2 v)^* B_1 u < 0. \tag{3.11}$$

Now, from (3.6) we obtain $B_2 v = -B_1 u - \theta p$ which substituted into (3.11) yields

$$\theta u^* B_1^T (-B_1 u - \theta p) + \bar{\theta}(-u^* B_1^T - \bar{\theta} p^*) B_1 u < 0,$$

or, equivalently,

$$-\theta u^* B_1^T B_1 u - \theta^2 u^* B_1^T p - \bar{\theta} u^* B_1^T B_1 u - \bar{\theta}^2 p^* B_1 u < 0.$$

Now, if $\Re(\theta) = 0$ then $\theta = \mathrm{i}\xi$ for some $\xi \in \mathbb{R}$, $\xi \neq 0$, where $\mathrm{i} = \sqrt{-1}$. After simplification, we find

$$-\theta u^* B_1^T B_1 u - \theta^2 u^* B_1^T p - \bar{\theta} u^* B_1^T B_1 u - \bar{\theta}^2 p^* B_1 u = \xi^2 \left(u^* B_1^T p + p^* B_1 u \right),$$

therefore condition (3.7) becomes

$$u^* B_1^T p + p^* B_1 u < 0$$

and since $u^* B_1^T p + p^* B_1 u = 2\Re(u^* B_1^T p)$, we conclude that

$$\Re(\theta) = 0 \Rightarrow \Re(u^* B_1^T p) < 0.$$

Likewise, from (3.6) we obtain $B_1 u = -B_2 v - \theta p$; substituting this into (3.11) and going through the same algebraic operations as before, we also find that

$$\Re(\theta) = 0 \Rightarrow \Re(v^* B_2^T p) < 0.$$

Therefore,

$$\Re((B_1 u + B_2 v)^* p) = \Re((u^* B_1^T + v^* B_2^T) p) < 0,$$

but together with (3.6) this implies $\Re((-\theta p)^* p) < 0$, or $\Re(\mathrm{i}\xi \|p\|_2^2) < 0$, which is clearly absurd since $\mathrm{i}\xi\|p\|_2^2$ is imaginary. This proves that $\mathcal{C}_\alpha$ cannot have purely imaginary eigenvalues. □

The restriction in Theorem 3.3 that A have positive definite symmetric part is not essential. If $A + A^T$ is only positive semidefinite (and singular), the alternating iteration (2.3)-(2.4) is still well defined. Moreover, the spectral radius of the iteration matrix cannot exceed 1. Indeed, the iteration matrix $\mathcal{T}_\alpha$ still satisfies $\varrho(\mathcal{T}_\alpha) \leq 1$, and if the symmetric part of A and B have no null vectors in common, the coefficient matrix $\mathcal{A}$ in (1.8) is still nonsingular; see again Lemma 1.1 in [4]. Hence, 1 is not an eigenvalue of $\mathcal{T}_\alpha = \mathcal{I} - \mathcal{P}_\alpha^{-1}\mathcal{A}$. However, it may happen that $\varrho(\mathcal{T}_\alpha) = 1$ for some choices of $\alpha > 0$. A simple example is given by

$$\mathcal{A} = \left[\begin{array}{cc|c} I & 0 & 0 \\ 0 & 0 & I \\ \hline 0 & -I & 0 \end{array}\right].$$

GMRES$(\mathcal{A}, \mathbf{x}_0, \mathbf{b}, \mathcal{P}_\alpha, m)$

Compute $\mathbf{r}_0 = \mathbf{b} - \mathcal{A}\mathbf{x}_0$, $\beta = \|\mathbf{r}_0\|_2$ and $\mathbf{v}_1 = \mathbf{r}_0/\beta$
for $j = 1, \ldots, m$ **do**
  Solve $\mathcal{P}_\alpha \mathbf{z} = \mathbf{v}_j$ for $\mathbf{z}$ by solving systems (2.3)-(2.4)
  Compute $\mathbf{w} = \mathcal{A}\mathbf{z}$
  for $i = 1, \ldots, j$ **do**
    $h_{i,j} = \langle \mathbf{w}, \mathbf{v}_i \rangle$
    $\mathbf{w} = \mathbf{w} - h_{i,j}\mathbf{v}_i$
  Compute $h_{j+1,j} = \|\mathbf{w}\|_2$ and $\mathbf{v}_{j+1} = \mathbf{w}/h_{j+1,j}$
  Define $V_m = [\mathbf{v}_1, \ldots, \mathbf{v}_m]$, $\bar{H}_m = \{h_{i,j}\}_{1 \le i \le j+1;\ 1 \le j \le m}$
Compute $\mathbf{y}_m = \text{argmin}_{\mathbf{y}} \|\beta \mathbf{e}_1 - \bar{H}_m \mathbf{y}\|_2$ and $\mathbf{x}_m = \mathbf{x}_0 + {\mathcal{P}_\alpha}^{-1} V_m \mathbf{y}_m$
If satisfied Stop, else set $\mathbf{x}_0 := \mathbf{x}_m$ and go to 1.

FIG. 4.1. *Preconditioned GMRES for solving system (1.8).*

Note that this matrix is nonsingular. It is easy to see that for $\alpha = 1$, the iteration matrix $\mathcal{T}_\alpha$ has only three distinct eigenvalues: $\lambda = 0$, $\lambda = \mathrm{i}$ and $\lambda = -\mathrm{i}$. Hence, the spectral radius is 1. Nevertheless, a simple modification of the basic algorithm yields a convergent iteration. To this end, recall that $\varrho(\mathcal{T}_\alpha) \le 1$ for all $\alpha > 0$; see (3.3). Let $\gamma \in (0, 1)$ be a parameter, then the matrix $(1-\gamma)\mathcal{I} + \gamma\mathcal{T}_\alpha$ has spectral radius less than 1 for all $\alpha > 0$. Indeed, the eigenvalues of $(1-\gamma)\mathcal{I} + \gamma\mathcal{T}_\alpha$ are of the form $1 - \gamma + \gamma\lambda$, where λ are the eigenvalues of $\mathcal{T}_\alpha$. It is easy to see that since $|\lambda| \le 1$ and $\lambda \ne 1$, all the quantities $1 - \gamma + \gamma\lambda$ have magnitude strictly less than 1. In practice, however, this modification is seldom used. In the next section we discuss Krylov subspace acceleration, which is much more effective and is applicable whether or not $\varrho(\mathcal{T}_\alpha) < 1$. Nevertheless, knowing that $\varrho(\mathcal{T}_\alpha) < 1$ is usueful because it implies that the spectrum of the preconditioned matrix lies entirely in the right half-plane, a desirable property for Krylov subspace acceleration. Moreover, the smaller is $\varrho(\mathcal{T}_\alpha)$, the more clustered the spectrum of the preconditioned matrix is around 1.

4. Krylov subspace acceleration. The basic method (2.3)-(2.4) although unconditionally convergent, is not competitive as a solver for problem (1.8), mainly due to the fact that convergence is generally slow. Fortunately, the rate of convergence can be greatly improved by Krylov subspace acceleration. In other words, $\mathcal{P}_\alpha$ can be used as a preconditioner for GMRES [23] or any other nonsymmetric Krylov method. It should be noted that when $\mathcal{P}_\alpha$ is used as a preconditioner, the pre-factor $\frac{1}{2\alpha}$ in (2.6) is irrelevant and can be neglected. In this paper we use the restarted GMRES algorithm with restart parameter m. The DS-preconditioned GMRES(m) algorithm is described in Fig. 4.1. Here preconditioning is applied on the right.

The rate of convergence of nonsymmetric Krylov iterations (like GMRES) preconditioned by $\mathcal{P}_\alpha$ depends on the particular choice of α. Finding the value of α that optimizes the rate of convergence appears to be a difficult problem in general. Indeed, in practice the convergence rate depends to a large extent on the size, shape, and location of the entire spectrum of the preconditioned matrix $\mathcal{P}_\alpha^{-1}\mathcal{A}$, and not just on the spectral radius of $\mathcal{T}_\alpha = \mathcal{I} - \mathcal{P}_\alpha^{-1}\mathcal{A}$. (The rate of convergence may also be affected by the conditioning of the eigenbasis of the preconditioned matrix, but this is usually difficult to estimate; see, e.g., [25, p. 17].) Numerical experiments (see below) suggest that there is a unique value α_* of α for which the number of preconditioned iterations is minimized, and this α_* is usually a small number ($0 < \alpha \ll 1$). Moreover, the convergence rate is not overly sensitive to small relative changes in α.

5. Implementation aspects. For the proposed approach to be successful, it is imperative that the action of the DS preconditioner (lines 3 and 10 in Fig. 1) be computed efficiently within each GMRES iteration. Written out explicitly, system (2.3) reads

$$\begin{bmatrix} A_1+\alpha I_{n_1} & 0 & B_1^T \\ 0 & \alpha I_{n_2} & 0 \\ -B_1 & 0 & \alpha I_{n_3} \end{bmatrix} \begin{bmatrix} u^{k+\frac{1}{2}} \\ v^{k+\frac{1}{2}} \\ p^{k+\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} \alpha u^k + f_1 \\ (\alpha I_{n_2} - A_2)v^k - B_2^T p^k + f_2 \\ B_2 v^k + \alpha p^k - g \end{bmatrix}, \quad (5.1)$$

while system (2.4) becomes

$$\begin{bmatrix} \alpha I_{n_1} & 0 & 0 \\ 0 & A_2+\alpha I_{n_2} & B_2^T \\ 0 & -B_2 & \alpha I_{n_3} \end{bmatrix} \begin{bmatrix} u^{k+1} \\ v^{k+1} \\ p^{k+1} \end{bmatrix} = \begin{bmatrix} (\alpha I_{n_1} - A_1)u^{k+\frac{1}{2}} - B_1^T p^{k+\frac{1}{2}} + f_1 \\ \alpha v^{k+\frac{1}{2}} + f_2 \\ B_1 u^{k+\frac{1}{2}} + \alpha p^{k+\frac{1}{2}} - g \end{bmatrix}. \quad (5.2)$$

Both systems (5.1) and (5.2) are highly reducible. Indeed, the second equation in (5.1) immediately yields $v^{k+\frac{1}{2}} = \frac{1}{\alpha}\left[(\alpha I_{n_2} - A_2)v^k - B_2^T p^k + f_2\right]$ together with the reduced system

$$\begin{bmatrix} A_1+\alpha I_{n_1} & B_1^T \\ -B_1 & \alpha I_{n_3} \end{bmatrix} \begin{bmatrix} u^{k+\frac{1}{2}} \\ p^{k+\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} \alpha u^k + f_1 \\ B_2 v^k + \alpha p^k - g \end{bmatrix}. \quad (5.3)$$

Likewise, system (5.2) is equivalent to $u^{k+1} = \frac{1}{\alpha}\left[(\alpha I_{n_1} - A_1)u^{k+\frac{1}{2}} - B_1^T p^{k+\frac{1}{2}} + f_1\right]$ together with the reduced system

$$\begin{bmatrix} A_2+\alpha I_{n_2} & B_2^T \\ -B_2 & \alpha I_{n_3} \end{bmatrix} \begin{bmatrix} v^{k+1} \\ p^{k+1} \end{bmatrix} = \begin{bmatrix} \alpha v^{k+\frac{1}{2}} + f_2 \\ B_1 u^{k+\frac{1}{2}} + \alpha p^{k+\frac{1}{2}} - g \end{bmatrix}. \quad (5.4)$$

Both systems (5.3) and (5.4) can be further reduced. Let $c^k = \alpha u^k + f_1$ and $d^k = B_2 v^k + \alpha p^k - g$. For (5.3), the second equation yields

$$p^{k+\frac{1}{2}} = \frac{1}{\alpha}\left(d^k + B_1 u^{k+\frac{1}{2}}\right) \quad (5.5)$$

which, substituted in the first one, yields

$$\left(A_1 + \alpha I_{n_1} + \frac{1}{\alpha} B_1^T B_1\right) u^{k+\frac{1}{2}} = c^k - \frac{1}{\alpha} B_1^T d^k. \quad (5.6)$$

Once this equation has been solved for $u^{k+\frac{1}{2}}$, we recover $p^{k+\frac{1}{2}}$ from (5.5). Similarly, let $c^{k+\frac{1}{2}} = \alpha v^{k+\frac{1}{2}} + f_2$ and $d^{k+\frac{1}{2}} = B_1 u^{k+\frac{1}{2}} + \alpha p^{k+\frac{1}{2}} - g$, then the second equation of (5.4) yields

$$p^{k+1} = \frac{1}{\alpha}\left(d^{k+\frac{1}{2}} + B_2 v^{k+1}\right) \quad (5.7)$$

which, substituted in the first one, yields

$$\left(A_2 + \alpha I_{n_2} + \frac{1}{\alpha} B_2^T B_2\right) v^{k+1} = c^{k+\frac{1}{2}} - \frac{1}{\alpha} B_2^T d^{k+\frac{1}{2}}. \quad (5.8)$$

Once this equation has been solved, the new value p^{k+1} of the pressure can be obtained from (5.7).

TABLE 6.1
GMRES iterations for Stokes problem for different grids with the optimal α and with $\alpha = 0.001$.

Grid	Its	α_{opt}	Its ($\alpha = 0.001$)
16×16	12	0.005	20
32×32	14	0.001	14
64×64	13	0.0005	16
128×128	13	0.0001	21

TABLE 6.2
GMRES iterations for Oseen problem for different grids and values of ν, optimal α.

Grid	$\nu = 0.1$	$\nu = 0.01$	$\nu = 0.001$
16×16	19	51	121
32×32	20	56	141
64×64	21	57	141
128×128	21	51	130

Hence, the bulk of the work in applying the preconditioner is in the solution of the two reduced systems (5.6) and (5.8). Each of these is a discrete analogue of a scalar, second-order, elliptic, anisotropic convection-diffusion-reaction equation. The anisotropy is in a sense artificial, since it depends on the size of the algorithmic parameter α: the smaller α is, the stronger the anisotropy in the diffusion terms. Note that for the Stokes and generalized Stokes problems, the convection terms are missing in (5.6) and (5.8) and the coefficient matrices are symmetric and positive definite. Some remarks on the solution of these two subsystems are given in the next section.

On the basis of the foregoing discussion, it is clear that the DS approach can be regarded as a *dimensionally segregated method*, i.e., a method where the values of the velocity components u and v (or u, v and w in 3D) are updated separately through a decoupling process; the new value of the pressure p is obtained at very low cost from the new velocity values.

6. Numerical experiments. In this section we report the results of numerical experiments on linear systems from Stokes and Oseen models of incompressible flow. We consider the leaky lid driven cavity problem discretized with Q2-Q1 finite elements on regular meshes of increasing size. All test problems were generated under Matlab using the IFISS software package [15] (see also [17]). We use the DS preconditioner in conjunction with restarted GMRES with $m = 20$ as the restart. In all cases the initial guess was the zero vector, and the stoping criterion was a reduction of at least six orders of magnitude of the initial residual norm. We discuss experiments for both steady and unsteady cases.

6.1. Steady problems. Application of the DS preconditioner requires, at each iteration, the solution of the linear systems (5.6) and (5.8), where the general form of A_i is given in (2.2). For the steady Stokes problems, $\sigma = 0$ and $N_i = 0$; hence, the coefficient matrices in these two systems are symmetric positive definite. The systems can be solved very efficiently with a sparse Cholesky factorization with an approximate minimum degree (AMD) ordering; see [1, 11]. The factorization is computed once and for all at the outset, and only forward and backward triangular solves need to be performed at each GMRES iteration. For the steady Oseen problem ($\sigma = 0$, $N_i \neq 0$ in (2.2)) the two systems (5.6) and (5.8) are nonsymmetric, although

TABLE 6.3
GMRES iterations for generalized Stokes ($\sigma = h^{-1}$) with the optimal α and with $\alpha = 0.001$.

Grid	Its	$\alpha_{\rm opt}$	Its ($\alpha = 0.001$)
16×16	12	0.005	20
32×32	14	0.001	14
64×64	13	0.0005	16
128×128	13	0.0001	21

TABLE 6.4
GMRES iterations for generalized Oseen ($\sigma = h^{-1}$) for different values of ν, optimal α.

Grid	$\nu = 0.1$	$\nu = 0.01$	$\nu = 0.001$
16×16	19	47	101
32×32	20	54	129
64×64	21	53	135
128×128	21	51	126

structurally symmetric. We compute sparse LU factorizations [11] using again an AMD reordering. These direct methods are much faster than iterative methods in the case of 2D problems; in the solution of large 3D problems iterative methods will have to be used instead, necessitating the use of a flexible Krylov method (like flexible GMRES [22]) for the outer iteration.

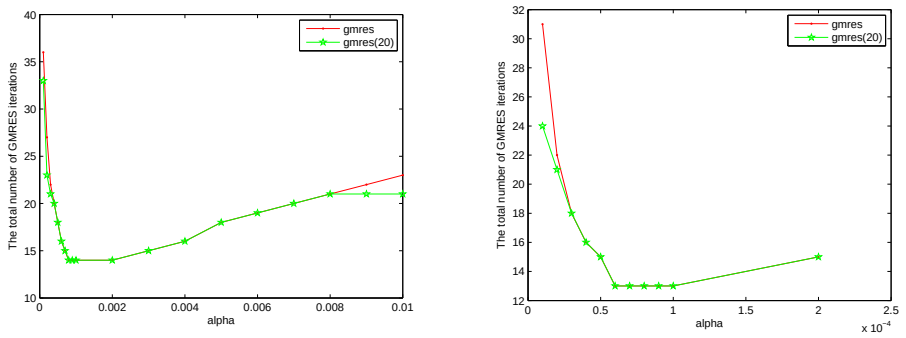
The first set of experiments is aimed at assessing the performance of the DS preconditioner on steady Stokes and Oseen problems, in particular to investigate the dependence on the discretization parameter h and on the viscosity ν.

In Table 6.1 we show iteration counts for DS-preconditioned GMRES(20) applied to the steady Stokes problem on a sequence of grids. We report results both for the optimal choice of α (determined experimentally) and for the fixed value $\alpha = 0.001$. We see that in both cases the iteration count is essentially independent of mesh size, with the suboptimal iteration count well within a factor of two from the optimal one. Our tests show that for the Stokes problem, small changes in the value of α do not have a dramatic effect on the number of iterations.

In Table 6.2 we report iteration count for the steady Oseen problem on a sequence of grids and for different values of ν, using optimal or near-optimal values of α. One can clearly see again that DS preconditioning results in h-independent convergence rates. There is, however, a noticeable deterioration in the rate of convergence as $\nu \to 0$.

6.2. Unsteady problems. Next, we report on analogous experiments involving the generalized Stokes problem (with $\nu = 1$) and the generalized Oseen problem (for several values of ν). A sequence of linear systems of this type needs to be solved when the time-dependent Stokes or Navier–Stokes equations are integrated numerically using fully implicit time-stepping schemes. Now the matrices A_1 and A_2 in (5.6)-(5.8) are of the form (2.2) where $\sigma = h^{-1}$ and M is the velocity mass matrix; also, $N_i = 0$ for generalized Stokes and $N_i \neq 0$ for generalized Oseen. As one can see from Tables 6.3-6.4, the results are virtually the same as those obtained in the steady case.

6.3. Choosing α. As with all parameter dependent preconditioners, some guidelines need to be provided for the choice of α. The analytic determination of the value



(a) 32×32 grid (b) 128×128 grid

FIG. 6.1. *Number of GMRES iterations vs. the value of α, unsteady Stokes problem.*

of α which results in the fastest convergence of the preconditioned GMRES iteration appears to be quite difficult, especially in the case of the Oseen problem. Our numerical experiments indicate that α should be taken very small; for the Q1-Q2 discretization used in our experiments, the best value of α is often of the order of 10^{-3} or even smaller. Moreover, the best value of α gets smaller as the mesh is refined; see Tables 6.1-6.3. Note that taking too small a value of α can lead to excessive ill-conditioning in the subsystems (5.6)-(5.8) to be solved at each GMRES iteration; in our experiments, however, this was never a problem.

Here we give two possible rules of thumb for the choice of α. One is to simply use a small fixed value of α, say $\alpha = 0.001$ or $\alpha = 0.0005$. Although not optimal, choosing α in this way yields nearly h-independent convergence rates which are fairly close to those obtained for the optimal choice of α. An advantage of this approach is that it makes the DS preconditioner essentially parameter-free, and therefore much easier to use.

The second possibility is to tie α to the discretization parameter h. Taking $\alpha \approx h^2$ gives a pretty accurate estimate of the optimal α in many cases, at least for h small enough.

Generally speaking, the rate of convergence of DS-preconditioned GMRES does not appear to be overly sensitive to the choice of α, in the sense that small relative changes in the size of α do not usually cause the number of iterations to change too drastically.

In Fig. 6.1 we show the total number of GMRES and GMRES(20) iterations for the solution of the unsteady Stokes problem as a function of α, for two choices of h. Note the different scales on the horizontal axes. From these plots, it appears that overestimating the parameter α does not lead to drastic changes in the number of iterations. Underestimating the optimal α can be more harmful, but this is easy to avoid.

6.4. Comparison with HSS preconditioning. It is interesting to compare these findings with some of the experimental results reported for GMRES with HSS preconditioning in [6]. The HSS preconditioner does not result in h-independence convergence in the case of steady problems (either Stokes or Oseen), although it tends to do so for unsteady problems with $\sigma = h^{-1}$. On the other hand, HSS preconditioning behaves better than DS in terms of the viscosity, in the sense that its rate of conver-

gence is not adversely affected as the viscosity decreases (in fact, it often converges faster as $\nu \to 0$.) Unfortunately, as discussed in [6], the efficient implementation of the HSS preconditioner appears to be very problematic for the convection form of the Navier–Stokes equations, whereas DS preconditioning does not have this limitation.

In summary, DS preconditioning appears to be superior to HSS preconditioning for steady Stokes and for both steady and unsteady Oseen problems in convection form. On the other hand, HSS preconditioning is well-suited for the rotation form of the equations, even for low viscosities. Since the (1,1) block in the rotation form of the Oseen problem is not block diagonal, the DS approach is not suitable for this class of problems. Finally, HSS and DS preconditioning exhibit similar good behavior on unsteady Stokes problems.

7. Conclusions and future work. In this paper we have introduced a "dimensional splitting" approach for the solution of saddle point systems in which the (1,1) block can be partitioned into a two-by-two block diagonal structure. Saddle point systems of this kind arise in a number of applications: in this paper we focused on linear systems arising from the discretization of two-dimensional incompressible flow problems.

We have established the convergence of the fixed-point iteration, and investigated experimentally its use as a preconditioner for restarted GMRES on a set of Stokes and Oseen problems (both steady and unsteady) discretized by Q2-Q1 finite elements. The numerical experiments indicate that the preconditioner results in h-independent convergence rates, with very fast convergence for the case of Stokes and generalized Stokes problems and for Oseen problems at moderate Reynolds numbers. As is the case for most preconditioners for the Oseen problem, the rate of convergence tends to worsen as the viscosity decreases.

Future work should include further analysis of the preconditioned iteration, including using Local Fourier Analysis [27, 3] for estimating the optimal value of the relaxation parameter α, and extension to the 3D case. We observe here that the basic alternating iteration (2.3)-(2.4) is of *Peaceman–Rachford type* and cannot be directly extended to the case of three splittings. Extension to the 3D case requires the alternating iteration to be of *Douglas–Rachford type*; see [26, pages 244–245]. From the viewpoint of implementation, the 3D case necessitates using (inexact) inner iterative solves for the subproblems that occur in the application of the preconditioner. The effect of inexact solves on the performance of the DS preconditioner needs to be investigated. We mention that a promising approach for solving systems of the type (5.6)-(5.8) was recently described in [9]. Another possibility would be to use the scalable algebraic multilevel solvers for scalar elliptic PDEs provided in the state-of-the-art Trilinos software package [19].

An interesting open question is whether the DS preconditioner can be modified so as to increase its robustness with respect to decreasing values of the viscosity, while at the same time maintaining h-independent convergence.

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