Two phase production

(i) For stability we assume that

$$\rho = \lambda \cdot \left(\frac{1}{\mu_1} + \frac{1}{\mu_2} \right) < 1;$$

note that ρ is the utilization rate of the machine. Balancing the flow between $\{(i-1,1),(i-1,2)\}$ and $\{(i,1),(i,2)\}$ yields

$$(p(i-1,1) + p(i-1,2))\lambda = p(i,2)\mu_2,$$

so

$$p(i,2) = \frac{\lambda}{\mu_2}(p(i-1,1) + p(i-1,2)). \tag{1}$$

Further, balancing the flow out of and into state $(n,2)$ gives

$$p(i,2)(\lambda + \mu_2) = p(i-1)\lambda + p(i,1)\mu_1,$$

and thus

$$p(i,1) = \frac{\lambda + \mu_2}{\mu_1} p(i,2) - \frac{\lambda}{\mu_1} p(i-1,2).$$

Substituting (1) into the equation above yields

$$p(i,1) = \frac{\lambda}{\mu_2} \left(\frac{\lambda + \mu_2}{\mu_1} p(i-1,1) + \frac{\lambda}{\mu_1} p(i-1,2) \right). \tag{2}$$

Hence, with (1) and (2) we find

$$p_i = (p(i,1), p(i,2)) = p_{i-1}R = \ldots = p_1 R^{i-1}, \qquad i \geq 1,$$

where

$$R = \frac{\lambda}{\mu_2} \begin{pmatrix} (\lambda + \mu_2)/\mu_1 & 1 \\ \lambda/\mu_1 & 1 \end{pmatrix}.$$

Finally, $p(0) = 1 - \rho$ and $p(1,2)$ and $p(1,1)$ follow from the balance equations in state 0 and $(1,2)$, respectively. This yields

$$p(1,2) = \frac{\lambda}{\mu_2} p(0) = \frac{\lambda}{\mu_2}(1-\rho), \qquad p(1,1) = \frac{\lambda + \mu_2}{\mu_1} p(1,2) = \frac{\lambda(\lambda + \mu_2)}{\mu_1 \mu_2}(1-\rho).$$

(ii) By PASTA we have

$$E(S) = (E(L) + 1) \cdot (\frac{1}{\mu_1} + \frac{1}{\mu_2}) - \rho_2 \cdot \frac{1}{\mu_1},$$

where $\rho_2 = \lambda/\mu_2$; the last term is subtracted, since, if the machine works on phase 2 on arrival, then the order does not have to wait for phase 1. Further, by Little's law,

$$E(L) = \lambda E(S).$$

Combining the two equations yields

$$E(S) = \frac{1}{1-\rho}\left(\frac{1}{\mu_1}(1-\rho_2) + \frac{1}{\mu_2}\right).$$

Alternatively, $E(L)$ can be obtained from the matrix-geometric solution, which gives

$$E(L) = \sum_{i=1}^{\infty} i p_i e = p_1 \sum_{i=1}^{\infty} i R^{i-1} e = p_1 (I - R)^{-2} e,$$

where e is the column vector of ones and I the indentity matrix. Then $E(S)$ follows form Little's law. For $\mu_1 = 6$, $\mu_2 = 30$ and $\lambda = 3$ we get $E(S) = 11/24$ hours.

(iii) Define a cycle as an idle period followed by a busy period. An idle period is exponential with mean $1/\lambda$ (by virtue of the memoryless property). Since the fraction of time the machine is working is equal to ρ, we get

$$\rho = \frac{E(\text{busy period})}{E(\text{idle period}) + E(\text{busy period})} = \frac{E(\text{busy period})}{1/\lambda + E(\text{busy period})}.$$

Hence,

$$E(\text{busy period}) = \frac{\rho/\lambda}{1-\rho} = \frac{1/\mu_1 + 1/\mu_2}{1-\rho},$$

and thus

$$E(\text{cycle}) = E(\text{idle period}) + E(\text{busy period}) = \frac{1/\lambda}{1-\rho} = \frac{5}{6} \text{ hours.}$$

So the average switch-on cost per hour is equal to

$$\frac{20}{E(\text{cycle})} = 24 \text{ dollar per hour.}$$

Alternatively, the average switch-on cost per hour is equal to

$$p(0) \cdot \lambda \cdot 20 = (1-\rho) \cdot \lambda \cdot 20 = \frac{6}{5} \cdot 20 = 24 \text{ dollar per hour.}$$

(iv) Replace state 0 by two states, $(0,1)$ and $(0,2)$. In the first state the system is empty and the machine is preparing phase 1; in $(0,2)$ the system is still empty, but the machine has completed phase 1.

(v) The production lead time of an order is now the same as in the system where phase 1 and 2 are interchanged (i.e. first phase 2 and then phase 1) and where an order does not have to wait for phase 1 (i.e. the order immediately leaves as soon as phase 2 has been completed). Hence, the mean production lead time is equal to the mean production lead time in (ii) minus the mean production time of phase 1,

$$E(S) = E(S(\text{ii})) - \frac{1}{\mu_1} = \frac{11}{24} - \frac{1}{6} = \frac{7}{24} \text{ hours.}$$

Alternatively, we may derive a relation for $E(S)$ by PASTA, yielding

$$E(S) = (E(L)+1) \cdot (\frac{1}{\mu_1} + \frac{1}{\mu_2}) - \rho_2 \cdot \frac{1}{\mu_1} - (1-\rho) \cdot \frac{1}{\mu_1},$$

where the last term is subtracted, since, if the machine is idle on arrival, then phase 1 has already been completed. Together with $E(L) = \lambda E(S)$ we obtain

$$E(S) = \frac{1}{1-\rho}\left(\frac{\rho_1}{\mu_1} + \frac{1}{\mu_2}\right) = \frac{7}{24} \text{ hours.}$$

(vi) The average swith-on cost per hour is the same as in (iii), so 24 dollar per hour.

Points

(i)	(ii)	(iii)	(iv)	(v)	(vi)
2	2	1	2	2	1