Unreliable machines

(i) The status of the machines (assuming there is always work) can be described by a Markov process with states i, $i = 0, 1, 2$, where i denotes the number of operational machines. Let ρ_i be the probability (or fraction of time) of being in state i. Balancing the flow between state 0 and 1 yields

$$\rho_0\theta = \rho_1\eta,$$

and similarly, balancing the flow between state 1 and 2,

$$\rho_1\theta = \rho_2 2\eta.$$

Together with the normalization $\rho_0 + \rho_1 + \rho_2 = 1$ we get

$$\rho_0 = \left[1 + \frac{\theta}{\eta} + \frac{\theta^2}{2\eta^2}\right]^{-1}, \qquad \rho_1 = \frac{\theta}{\eta}\,\rho_0, \qquad \rho_2 = \frac{\theta^2}{2\eta^2}\,\rho_0.$$

(ii) The maximal throughput of the machines is $\rho_0 \cdot 0 + \rho_1 \cdot \mu + \rho_2 \cdot 2\mu$, which should be greater than the inflow. So for stability we have to require

$$\rho_1 \cdot \mu + \rho_2 \cdot 2\mu > \lambda.$$

(iii) The transition rate diagram is shown below.

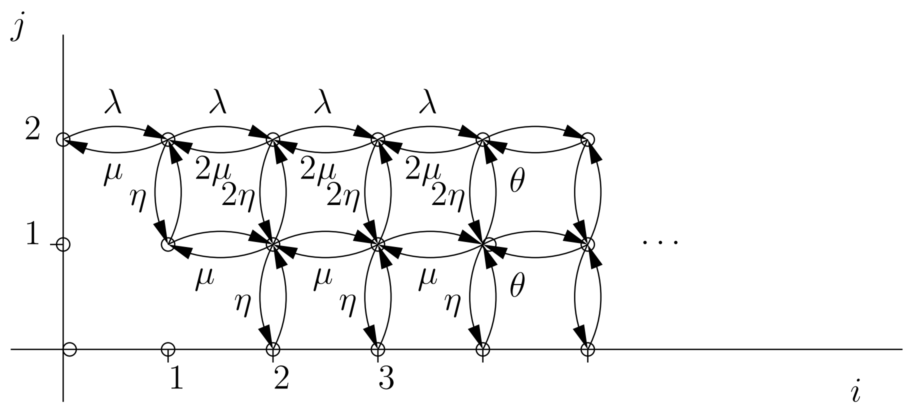

Figure 1: Transition-rate diagram for the model with two machines subject to operational failures and a single repair man

The balance equations in states (i, j) with $i \geq 2$ are given by

$$\begin{aligned}
p(i,0)(\lambda+\theta) &= p(i-1,0)\lambda + p(i,1)\eta, \\
p(i,1)(\lambda+\mu+\theta+\eta) &= p(i-1,1)\lambda + p(i+1,1)\mu + p(i,0)\theta + p(i,2)2\eta, \\
p(i,2)(\lambda+2\mu+2\eta) &= p(i-1,2)\lambda + p(i+1,2)2\mu + p(i,1)\theta,
\end{aligned}$$

where $p(1,0) = 0$ by convention. For the balance equations in the boundary states $(0,2)$, $(1,1)$ and $(1,2)$ we get

$$\begin{aligned} p(0,2)\lambda &= p(1,2)\mu, & (1)\\ p(1,2)(\lambda+\mu+\eta) &= p(0,2)\lambda + p(2,2)2\mu + p(1,1)\theta, & (2)\\ p(1,1)(\lambda+\theta) &= p(2,1)\mu + p(1,2)\eta. & (3) \end{aligned}$$

Substitution of the trial solution

$$p(i,j) = x^i y(j), \qquad i = 1,2,\ldots, j = 0,1,2,$$

in the balance equations for states (i,j) with $i \geq 2$, we find

$$yA(x) = 0, \tag{4}$$

where $y = (y(0), y(1), y(2))$ and

$$A(x) = \begin{pmatrix} \lambda - x(\lambda+\theta) & x\theta & 0 \\ x\eta & \lambda + x^2\mu - x(\lambda+\mu+\theta+\eta) & x\theta \\ 0 & x2\eta & \lambda + x^2 2\mu - x(\lambda+2\mu+2\eta) \end{pmatrix}.$$

Equation (4) has a non-null solution for y if $\det(A(x)) = 0$. Provided the system is stable, it can be shown that the determinantal equation has exactly three roots x with $|x| < 1$. Label these roots x_0, x_1, x_2 and let y_j be a non-null solution of (4) with $x = x_j$, $j = 0,1,2$. We set

$$p_i = (p(i,0), p(i,1), p(i,2)) = \sum_{j=0}^{2} c_j y_j x_j^{i-1}, \qquad i = 1,2,\ldots,$$

and then determine the unknown coefficients c_0, c_1 and c_2 from the boundary equations and the normalization equation. Substitution of this linear combination in (3) yields

$$\sum_{j=0}^{2} c_j x_j \left(y_j(1)(\lambda+\theta-x_j\mu) - y_j(2)\eta\right) = 0,$$

and the convention $p(1,0) = 0$ gives

$$\sum_{j=0}^{2} c_j x_j y_j(0) = 0.$$

From the normalization equation we get

$$\begin{aligned} 1 &= p(0,2) + \sum_{i=1}^{\infty}(p(i,0) + p(i,1) + p(i,2)) \\ &= \sum_{j=0}^{2} c_j x_j y_j(2)\frac{\mu}{\lambda} + \sum_{j=0}^{2} c_j \frac{x_j}{1-x_j}(y_j(0) + y_j(1) + y_j(2)), \end{aligned}$$

where the expression for $p(0,2)$ follows from (1). The coefficients c_0, c_1 and c_2 can now be solved from the above three equations.

(iv) Let L^q denote the number of jobs waiting in the queue. Then we have

$$E(L^q) = \sum_{i=2}^{\infty}(i-2)(p(i,0)+p(i,1)+p(i,2)) = \sum_{j=0}^{2} c_j \frac{x_j^3}{(1-x_j)^2}(y_j(0)+y_j(1)+y_j(2)),$$

and $E(W) = E(L^q)/\lambda$ by Little's law. For the mean total number of jobs in the system, $E(L)$, we find

$$E(L) = \sum_{i=1}^{\infty} i(p(i,0)+p(i,1)+p(i,2)) = \sum_{j=0}^{2} c_j \frac{x_j}{(1-x_j)^2}(y_j(0)+y_j(1)+y_j(2)),$$

from which $E(S)$ follows by Little's law. The mean number of machines waiting for repair is given by

$$E(L_M^q) = \sum_{i=2}^{\infty} p(i,0) = \sum_{j=0}^{2} c_j \frac{x_j^2}{1-x_j} y_j(0),$$

and the utilization of the repair man by

$$\rho_R = \sum_{i=1}^{\infty}(p(i,0)+p(i,1)) = \sum_{j=0}^{2} c_j \frac{x_j}{1-x_j}(y_j(0)+y_j(1)).$$

Finally, the mean waiting time for repair follows from Little's law, i.e.

$$E(W_M) = \frac{E(L_M^q)}{\rho_R \theta}.$$

(v) Some of the performance characteristics in (iv) are listed in the table below for $\mu = 1$ and $\eta/\theta = 0.1$.

λ	η	$E(W)$	$E(S)$	$E(L_M^q)$
1.5	1	2.52	3.63	0.012
	0.025	3.62	4.73	0.013
1.7	1	8.96	10.1	0.015
	0.025	13.0	14.1	0.015

Points

(i)	(ii)	(iii)	(iv)	(v)
2	1	3	2	2