Waiting time analysis of a $U/PH/1$ system

(i) Suppose the processing time X is given by

$$X = \begin{cases} X_1 & \text{w.p. } p, \\ X_1 + X_2 & \text{w.p. } (1-p). \end{cases}$$

Then

$$E(X) = \frac{2-p}{\mu}, \qquad \text{var}(X) = \frac{2-p^2}{\mu^2}, \qquad c_X^2 = \frac{\text{var}(X)}{E^2(X)} = \frac{2-p^2}{(2-p)^2}.$$

Note that c_X^2 is increasing in p for $0 \le p \le 1$, $c_X^2 = 1/2$ for $p = 0$ and $c_X^2 = 1$ for $p = 1$. In this situation the squared coeffcient of the production time $c_X^2 = 9/16$, so between $1/2$ and 1. If we determine p such that

$$c_X^2 = \frac{9}{16} = \frac{2-p^2}{(2-p)^2},$$

we find $p = 0.12133$. The parameter μ follows from

$$E(X) = 4 = \frac{2-p}{\mu},$$

yielding $\mu = 0.46967$.

(ii) The balance equations for p_n are given by

$$p_n = p \sum_{i=0}^{\infty} p_{n-1+i} a_i + (1-p) \sum_{i=0}^{\infty} p_{n-2+i} a_i,$$

where $p_{-1} = 0$ by convention, and

$$a_i = \int_4^8 \frac{(\mu t)^i}{i!} e^{-\mu t} \frac{dt}{4}, \qquad i = 0, 1, 2, \ldots$$

The solution is of the form

$$p_n = c_1(1-\sigma_1)\sigma_1^n + c_2(1-\sigma_2)\sigma_2^n, \qquad n = 0, 1, 2, \ldots$$

where σ_1 and σ_2 are the roots in $(-1, 1)$ of

$$\begin{aligned} \sigma^2 &= (p\sigma + 1 - p) E(e^{-\mu(1-\sigma)A}) \\ &= (p\sigma + 1 - p) \int_4^8 e^{-\mu(1-\sigma)t} \frac{dt}{4} \\ &= (p\sigma + 1 - p) \frac{e^{-4\mu(1-\sigma)} - e^{-8\mu(1-\sigma)}}{4\mu(1-\sigma)}, \end{aligned}$$

where the random variable A denotes the interarrival time. The coefficients c_1 and c_2 are given by

$$c_1 = \frac{1 - 1/\sigma_2}{1/\sigma_1 - 1/\sigma_2}, \qquad c_2 = \frac{1 - 1/\sigma_1}{1/\sigma_2 - 1/\sigma_1}.$$

Numerical soution of the equation for σ yields

$$\sigma_1 = 0.46396, \qquad \sigma_2 = -0.19094,$$

and thus

$$c_1 = 0.84372, \qquad c_2 = 0.15628.$$

(iii) The mean waiting time is

$$E(W) = \frac{c_1\sigma_1}{\mu(1-\sigma_1)} + \frac{c_2\sigma_2}{\mu(1-\sigma_2)} = 1.5015 \text{ min.}$$

(iv) The complementary waiting time distribution is given by

$$P(W > t) = c_1\sigma_1 e^{-\mu(1-\sigma_1)t} + c_2\sigma_1 e^{-\mu(1-\sigma_2)t}, \qquad t \geq 0.$$

Hence, for $t = 10$ we have

$$P(W > 10) = 0.03146.$$

Points

(i)	(ii)	(iii)	(iv)
3	3	2	2