CS171 Introduction to Computer Science II

Recursion

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Announcement

- Hw3 is extended to Monday
- Quiz 1 will be distributed Tuesday

Recursion

- Recursion concept
- Examples
  - Factorial
  - Fibonacci
  - GCD
  - Recursive graph Htree
- More examples
  - Binary search
  - Tower of Hanoi
- Cost analysis of recursive algorithms
- Divide and conquer

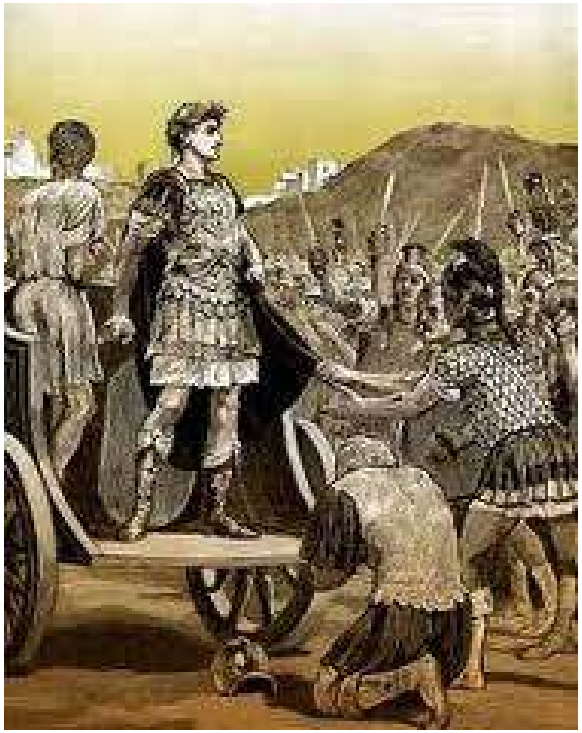
Recursive Method

- A method that calls itself (direct recursion)
- Every recursive method must have a base case that is not recursive

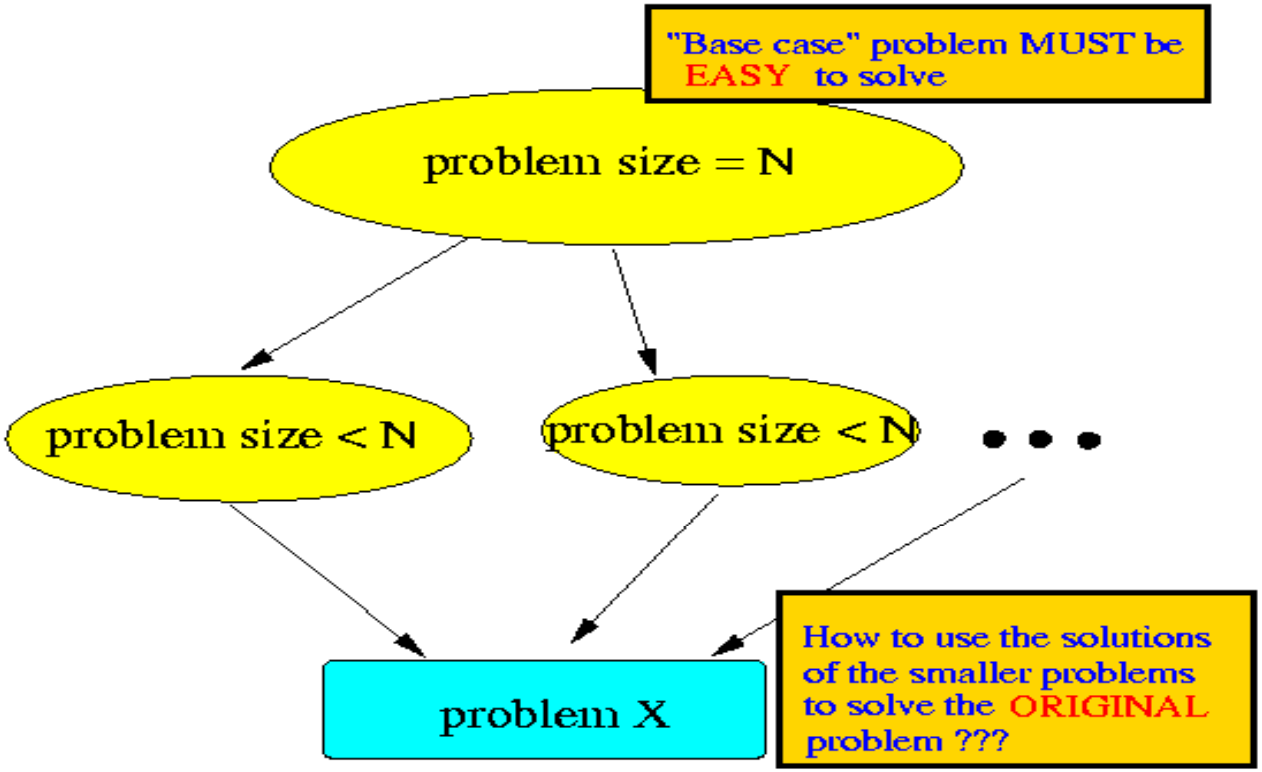
```
void recursiveMethod() {
     …
     if (base case) {
            … …
     }
     else {
            … …
            recursiveMethod();
            … …
     }
 }
```

Divide and Conquer

- A big problem is divided into two or more problems with smaller sizes (**sub-problems**).
- To solve a sub-problem, you **again** divide it into even smaller problems.
- The process continues until you get to the **base case**, which can be solved trivially.



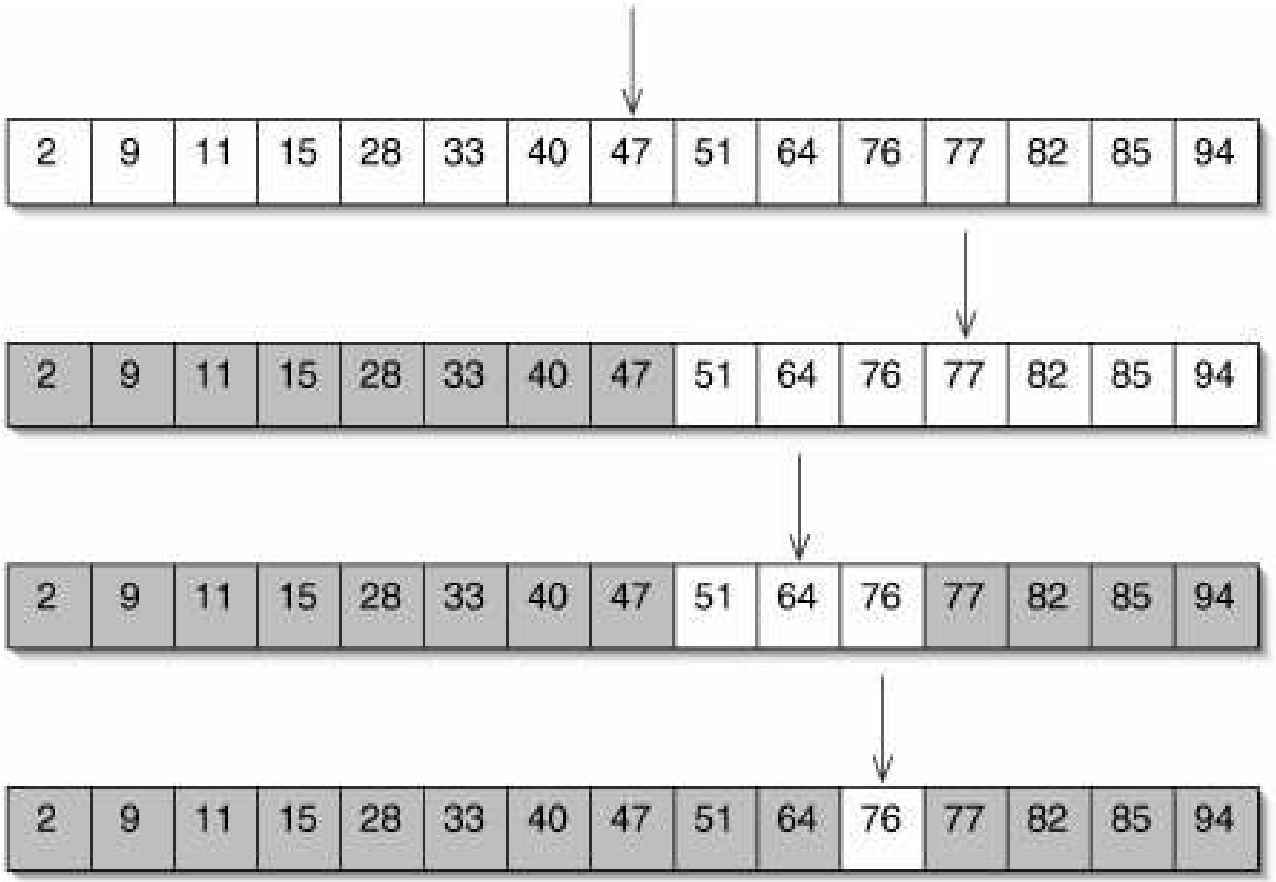
Recursion as a problem solving



- Solve one or more problems that are identical to the original problem but with smaller size
- Solve original problem by solutions of smaller problems

Binary Search in Ordered Array

- Compares the middle element with search key and reduces the search range by half in each iteration



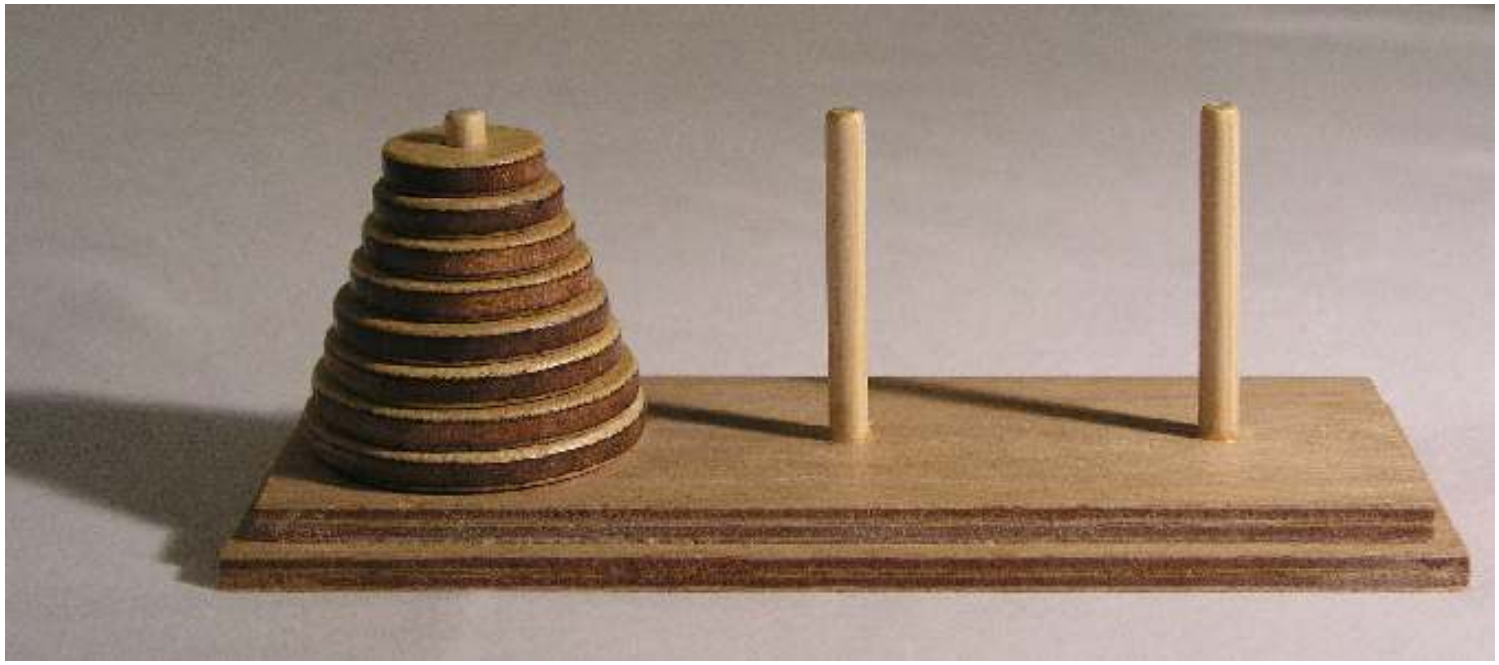
Binary Search – Recursive solution

```
public int find(long searchKey) {
      return recFind(searchKey, 0, nElems-1);
}
private int recFind(long searchKey, int lowerBound,
                    int upperBound){
      int curIn;
      curIn = (lowerBound + upperBound ) / 2;
      if(a[curIn]==searchKey)                    // found it
         return curIn;
      else if(lowerBound > upperBound) // can't find it
         return nElems;
      else if(a[curIn] < searchKey)// upper half
         return recFind(searchKey, curIn+1, upperBound);
      else                                  // lower half
         return recFind(searchKey, lowerBound, curIn-1);
 }
```

Recursion vs. Iteration

- Recursion: calls itself one or more times until a condition is met
- Iteration: repeating for a specified number of times or until a condition is met
- **Every recursive function** can be **transformed** to an **iterative function** using a **stack** and **vice versa**
- Recursion is an **elegant** way to solve many practical problems but usually **sacrifices memory** and **computational efficiency** (what is the overhead?)

The Towers of Hanoi

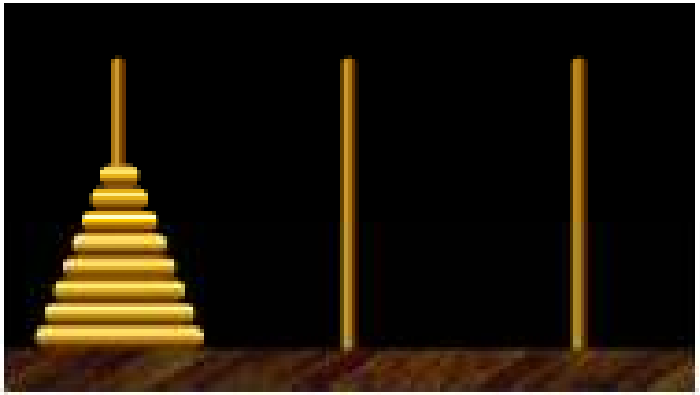


A **B** **C**

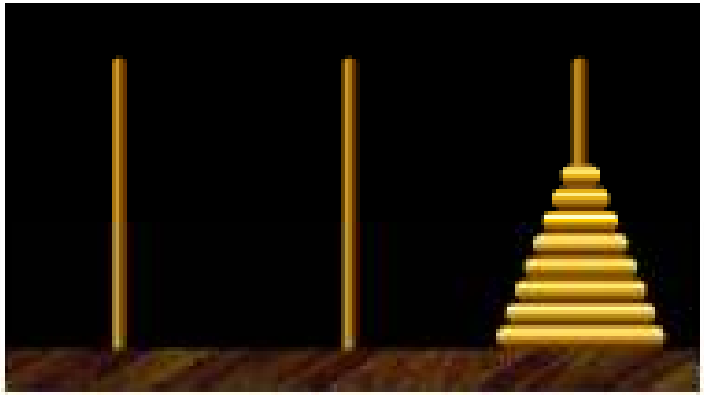
The Towers of Hanoi

Move all the discs from the leftmost peg to the rightmost one.

- Only one disc may be moved at a time.
- A disc can be placed either on empty peg or on top of a larger disc.



Start



Finish

Is world going to end (according to legend)?

- 40 golden discs on 3 diamond pegs.
- World ends when certain group of monks accomplish task.

Edouard Lucas (1883)

The Towers of Hanoi

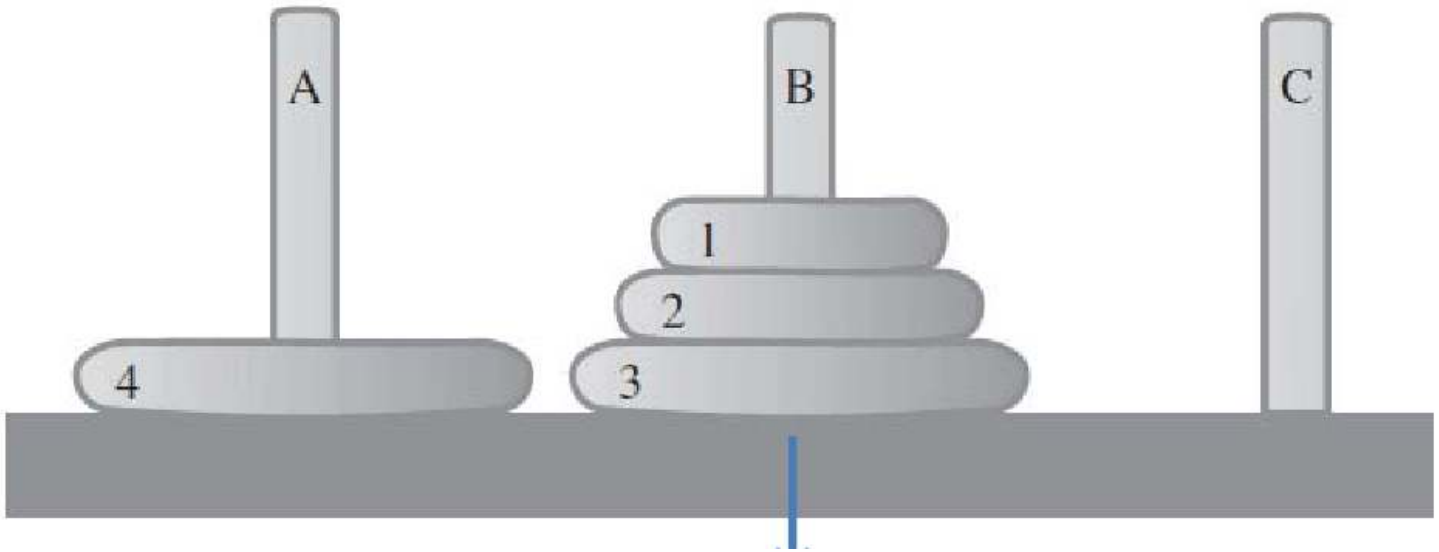
- How do we move the disks to achieve the goal?

- We also want to know in general, how many steps it takes to move N disks.
- Let's play with it to get some intuition:
  - N=1, N=2, N=3, N=4

http://www.mazeworks.com/hanoi/

The Towers of Hanoi

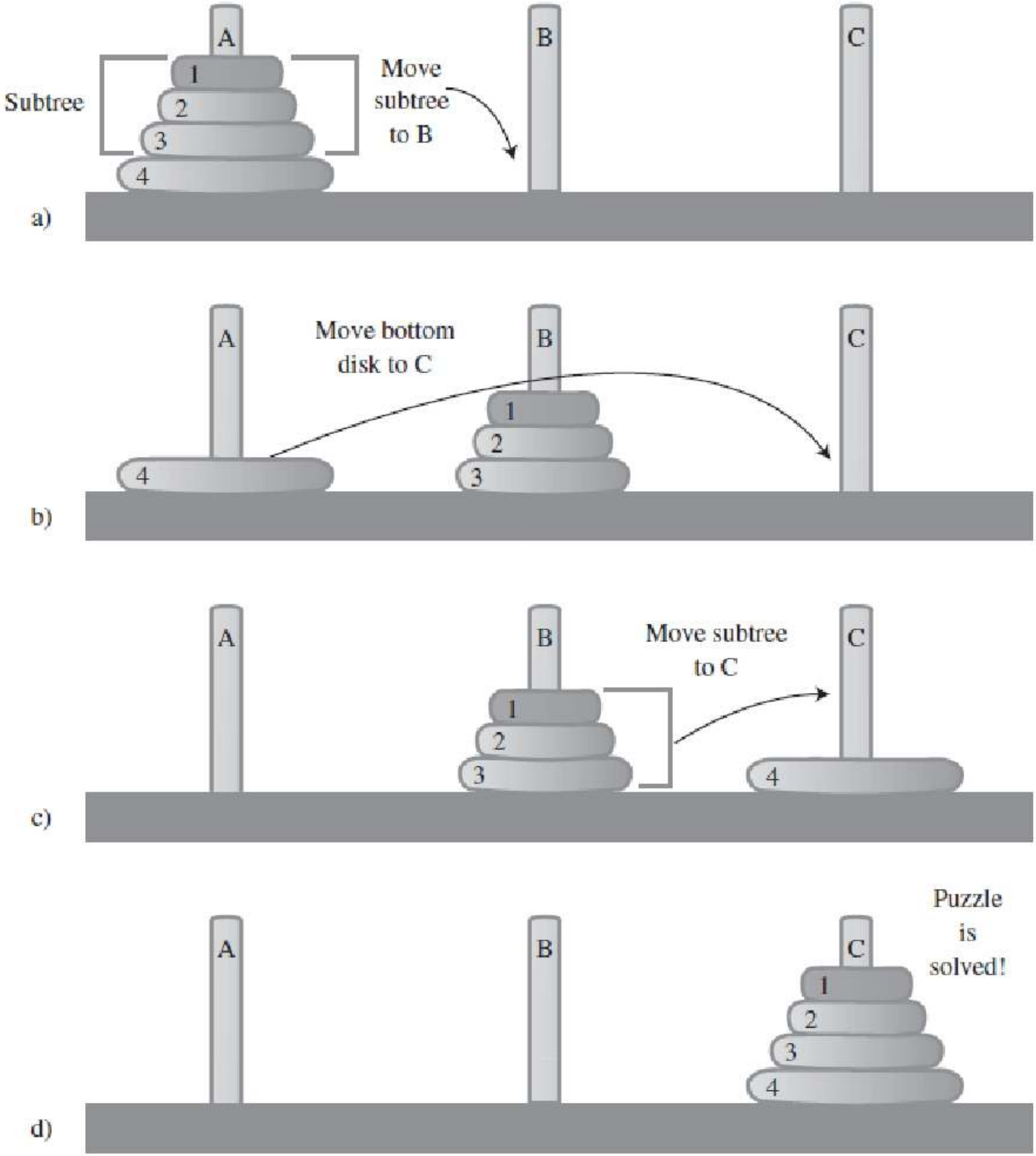
- Let's call the initial pyramid-shaped arrangements of disks on column A a ***tree***.
- We call a smaller set of the disks a ***subtree***.
- It turns out that the intermediate steps in the solution involves moving a subtree.



The Towers of Hanoi

- Idea:
  - Assume, for now, that you have a (magical) way of moving a subtree **from A to B via C**;
  - Then you move A to C;
  - Finally move the subtree **from B to C via A**.
  - This is similar to the 2-disk case, except the top disk is now a subtree.

- How can we move the subtree?
- What's the base case?


A
B
C
Subtree
1
2
3
4
Move
subtree
to B
a)
A
B
C
Move bottom
disk to C
1
2
3
4
b)
A
B
C
Move subtree
to C
1
2
3
4
c)
A
B
C
Puzzle
is
solved!
1
2
3
4
d)

The Towers of Hanoi

- Suppose you want to move n disks from a **source tower S** to a **destination tower D**, via an **intermediate tower I**.
- Initially
  - Source tower is A
  - Destination tower is C
  - Intermediate tower is B

The Towers of Hanoi

1. Move the subtree consisting of the top n-1 disks from S to I;
2. Move the remaining (largest) disk from S to D;
3. Move the subtree from I to D.

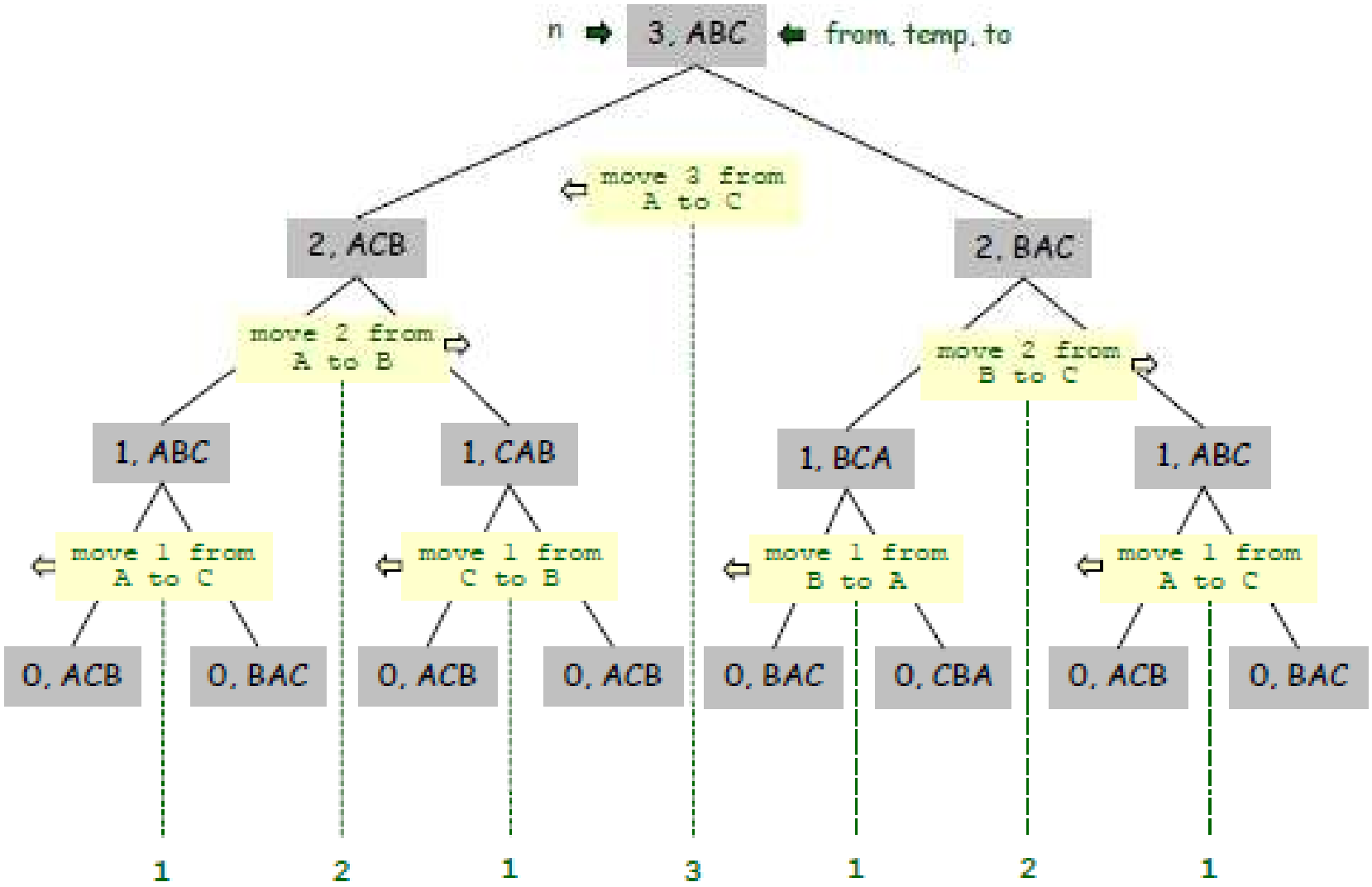
Towers of Hanoi: Implementation

```
public class Hanoi {

   static void hanoi(int n, char from, char temp, char to) {
      if (n == 0) return;
      hanoi(n-1, from, to, temp);
      System.out.println("Move disc " + n);
      System.out.println(" from " + from + " to " + to);
      hanoi(n-1, temp, from, to);
   }

   public static void main(String[] args) {
      int N = Integer.parseInt(args[0]);
      hanoi(N, 'A', 'B', 'C');
   }
}
```

General Approach: Recursion Tree



The Tower of Hanoi

- How many steps does the solution take to solve a N-disk problem?
- N=1:
- N=2:
- N=3:
- N=4:

- When is the world going to end (N=64)?

The Tower of Hanoi

- How many steps does the solution take to solve a N-disk problem?
- N=1: 1 step
- N=2: 3 steps
- N=3: 7 steps
- N=4: 15 steps

- When is the world going to end (N=64)?

Using Recurrence Relation for Cost Analysis

- Define the cost function T(N) using recurrence relation and define base cases
- Solve the recurrence relation
- Derive Big O function

Defining **Recurrence Relation**

- A recurrence relation is an equation that recursively defines a sequence: each term of the sequence is defined as a function of the preceding terms
- Examples:
  - $T(n) = T(n-1) + 1$

Solving Recurrence Relations

- Rewrite T(N), T(N-1), T(N-2), …, with the recurrence formula
- Discover the patterns and find an expression
- Check the correctness
  - Substitute solution in initial conditions
  - Substitute solutions in the recurrence relation

Example

- Loop

```
for (i=0; i<N; i++) {
  do_something();
}
```

- Recurrence relation
- $T(n) = T(n-1) + 1$
- $T(1) = 1$

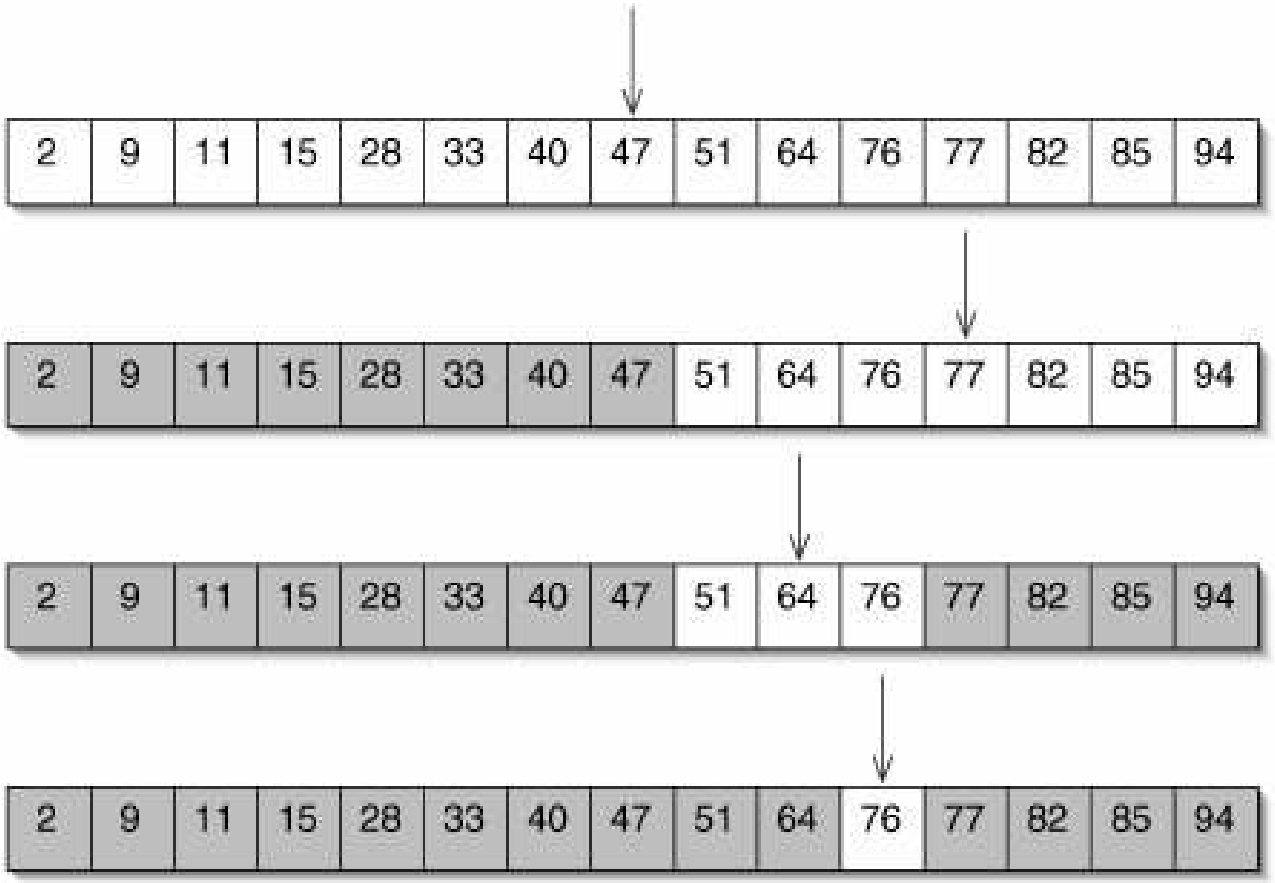
Example (cont'd)

- $T(n) = T(n-1) + 1$
- $T(1) = 1$

- Expansion
  - $T(n) = T(n-1) + 1 = T(n-2) + 1 + 1 = T(n-3) + 1 + 1 + 1 = \ldots$
- Discover pattern and solution
  - $T(n) = T(1) + n - 1 = n$
- Verification
  - $T(1) = 1$
  - $T(n) = T(n-1) + 1$

Binary Search Example

- Binary Search Cost Function
- T(n) = T(n/2) + 1
- T(1) = 1



Binary Search Example: Solution

- Binary Search Cost Function
- $T(n) = T(n/2) + 1$
- $T(1) = 1$

- $N=2^k$
- $T(2^k) = T(2^{k-1}) + 1 = T(2^{k-2}) + 1 + 1 + \ldots$
- $T(2^k) = T(2^0) + k = 1 + k$
- $T(N) = 1 + \lg N$

The Tower of Hanoi

- How many steps does the solution take to solve a N-disk problem?
- Use recursive formula

  T(N) = T(N-1) + 1 + T(N-1) = 2*T(N-1) + 1
- So how to solve this?

$$T(N) = 2^{N\text{-}1} + 2^{N-2} + 2^{N-3} + \ldots + 1 = 2^{N} - 1$$

- So it takes an exponential number of steps!

When is the world going to end?

- Takes 585 billion years for N = 64 (at rate of 1 disc per second).

Summary

- Recursion: powerful tool allows for elegant solutions
  - But, computational overhead is high
- Can analyze runtime:
  - Intuition: draw recursive call tree
  - Analysis: **Recurrence** relations