CS171 Introduction to Computer Science II

Recursion (cont.) + MergeSort

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Reminders

- Hw3 due yesterday (use late credit if needed)
- Hw4 due Friday

Today

- Recursion (cont.)
  - Concept and examples
  - Analyzing cost of recursive algorithms
  - Divide and conquer
  - Dynamic programming

- MergeSort

Fibonacci Numbers

- Recursive formula:

$$F(n) = F(n-1) + F(n-2)$$

$$F(0) = 0, \;\; F(1) = 1$$

- 0, 1, 1, 2, 3, 5, 8, 13, ...

Fibonacci Numbers

```
int F(int n)
{
    if (n==0)
        return 0;
    else if (n==1)
        return 1;
    else
        return F(n-1)+F(n-2);
}
```

Consider F(5), how is it computed?

Runtime of Recursive Fibonacci

```
                 F(5)
               /      \
          F(4)            F(3)
         /    \          /    \
     F(3) F(2)       F(2) F(1)
    /    \
F(2) F(1)
```

Dynamic programming

- **Dynamically** solve a smaller problem
  - Solve each small problem only once
- Applicable when
  - Overlapping subproblems are slightly smaller (vs. divide and conquer)
  - *Optimal substructure:* the solution to a given optimization problem can be obtained by the combination of optimal solutions to its subproblems.

Memoization

- A technique for dynamic programming
  - A memoized function "remembers" the results corresponding to some set of specific inputs.
  - Subsequent calls with remembered inputs return the remembered result, rather than recomputing it
- General structure

```
static int sol[]; //save solutions for each problem
static … recursiveFunc(int N) {
   if (sol[N] is available)
         return sol[N];
  …
  similar to regular recursion
         except: saving the solution sol[N]
}
```

Fibonacci with Dynamic Programming

```
static int sol[];
static int F(int n) {
   if (sol[n] > 0)    //pre-computed already
        return sol[n];

   if (n==0) {
        sol[n] = 1;
        return 1;
   }
   else if (n==1) {
        sol[n] = 1;
        return 1;
   }
   else {
        sol[n] = F(n-1) + F(n-2);
        return sol[n];
   }
}
```

Example: F(5)

Today

- Recursion (cont.)
  - Concept and examples
  - Analyzing cost of recursive algorithms
  - Divide and conquer
  - Dynamic programming

- MergeSort

Advanced Sorting

- We've learned some simple sorting methods, which all have quadratic costs.
  - Easy to implement but slow.

- Much faster advanced sorting methods:
  - **Merge Sort**
  - Quick Sort
  - Radix Sort

MergeSort

- Basic idea
  - Divide array in half
  - Sort each half (how?)
  - Merge the two sorted halves

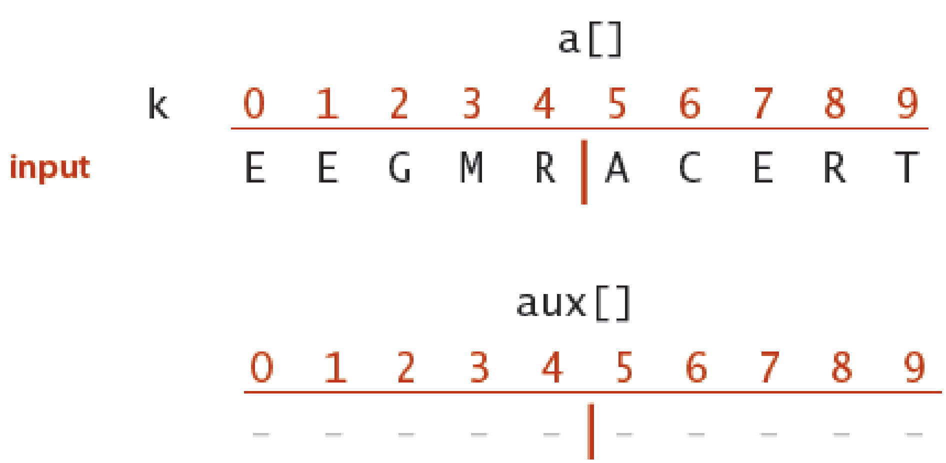
Mergesort overview

Merge Sort

- This is a divide and conquer approach:
  - Partition the original problem into two sub-problems;
  - Use recursion to solve each sub-problem;
  - Sub-problem eventually reduces to base case;
  - The results are then combined to solve the original problem.

Merge Two Sorted Arrays

- A **key step** in mergesort
- Assume arrays A (left half) and B (right half) are **already sorted**.
- Merge them to array C (the original array), such as C contains all elements from A and B, and remains sorted
- Use an auxiliary array aux[]
- Example on board and demo



Merging Two Sorted Arrays

1. Start from the **first** elements of A and B;
2. Compare and copy the **smaller** element to C;
3. Increment indices, and continue;
4. If reaching the end of either A or B, quit loop;
5. If either A (or B) contains **remaining** elements, append them to C.

```
private static void merge(Comparable[] a, Comparable[] aux, int lo, int mid, int hi)
{
   assert isSorted(a, lo, mid);    // precondition: a[lo..mid]   sorted
   assert isSorted(a, mid+1, hi);  // precondition: a[mid+1..hi] sorted

   for (int k = lo; k <= hi; k++)                                   copy
      aux[k] = a[k];

   int i = lo, j = mid+1;                                           merge
   for (int k = lo; k <= hi; k++)
   {
      if      (i > mid)              a[k] = aux[j++];
      else if (j > hi)               a[k] = aux[i++];
      else if (less(aux[j], aux[i])) a[k] = aux[j++];
      else                           a[k] = aux[i++];
   }

   assert isSorted(a, lo, hi);     // postcondition: a[lo..hi] sorted
}
```

Merging Two Sorted Arrays: Analysis

- How many comparisons is required?
- How many copies?

Merging Two Sorted Arrays (Sol.)

- How many comparisons is required?

  at most (A.length + B.length)
- How many copies?

  A.length + B.length

Assertions

Assertion. Statement to test assumptions about your program.

- Helps detect logic bugs.
- Documents code.

Java assert statement. Throws an exception unless boolean condition is true.

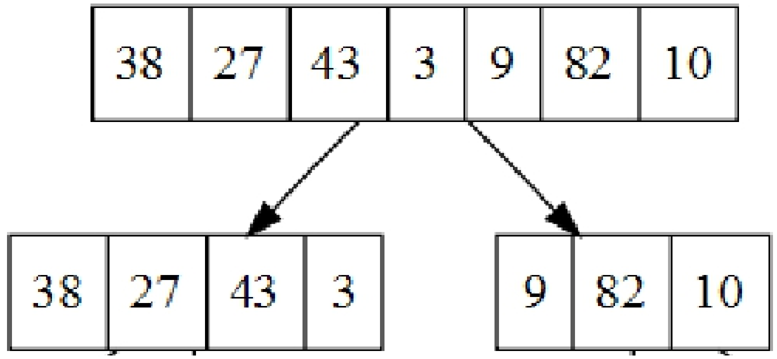
```
assert isSorted(a, lo, hi);
```

Can enable or disable at runtime. ⇒ No cost in production code.

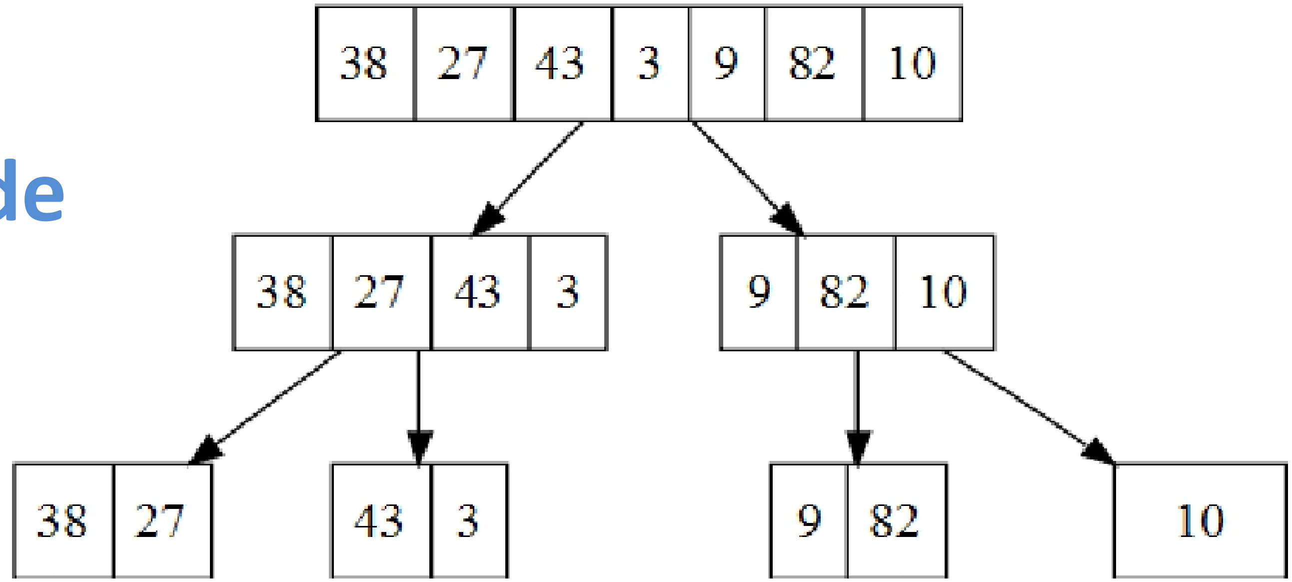
```
java -ea MyProgram   // enable assertions
java -da MyProgram   // disable assertions (default)
```

Best practices. Use to check internal invariants. Assume assertions will be disabled in production code (e.g., don't use for external argument-checking).

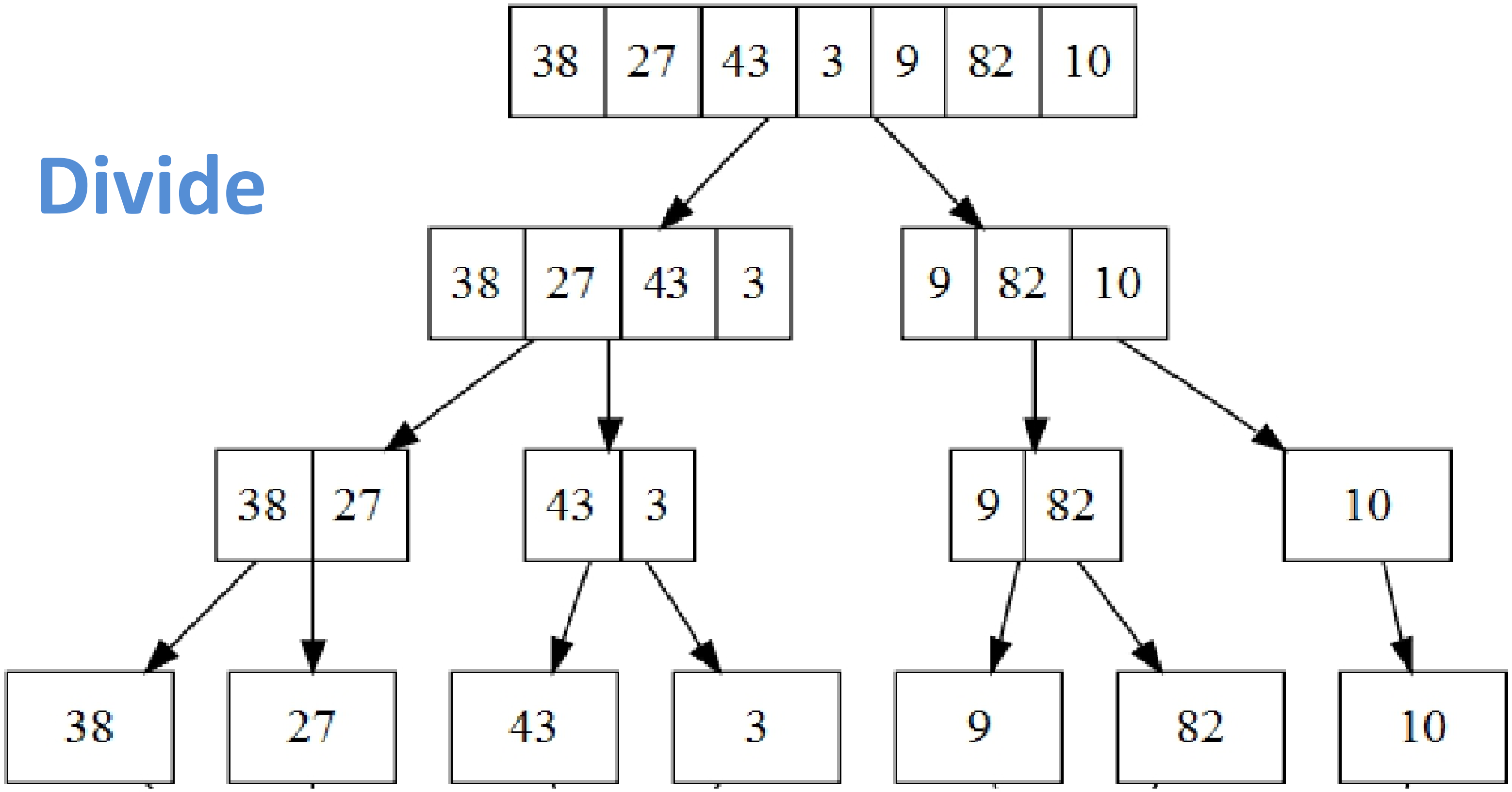
Divide



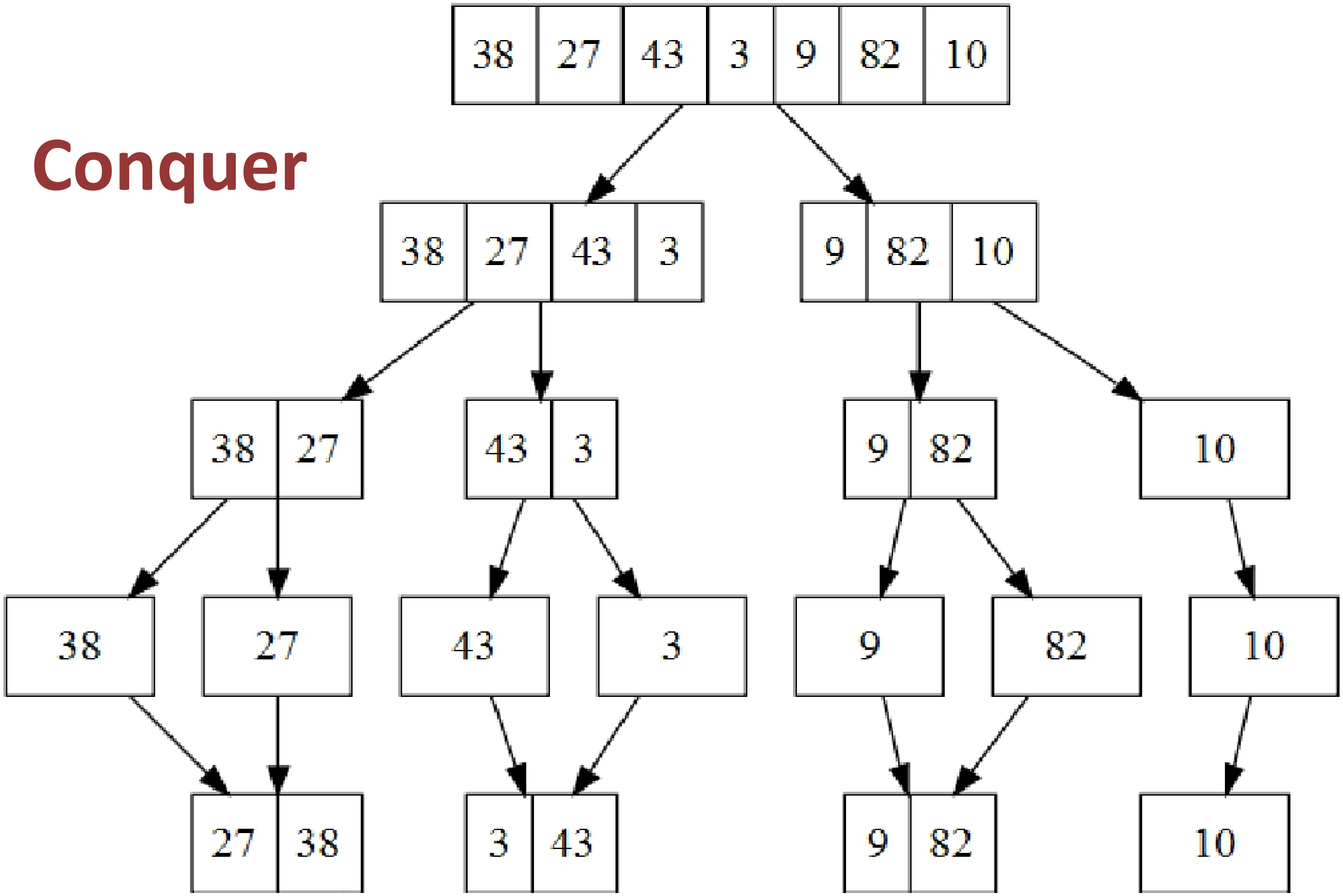
Divide



Divide



Conquer


38 27 43 3 9 82 10
38 27 43 3
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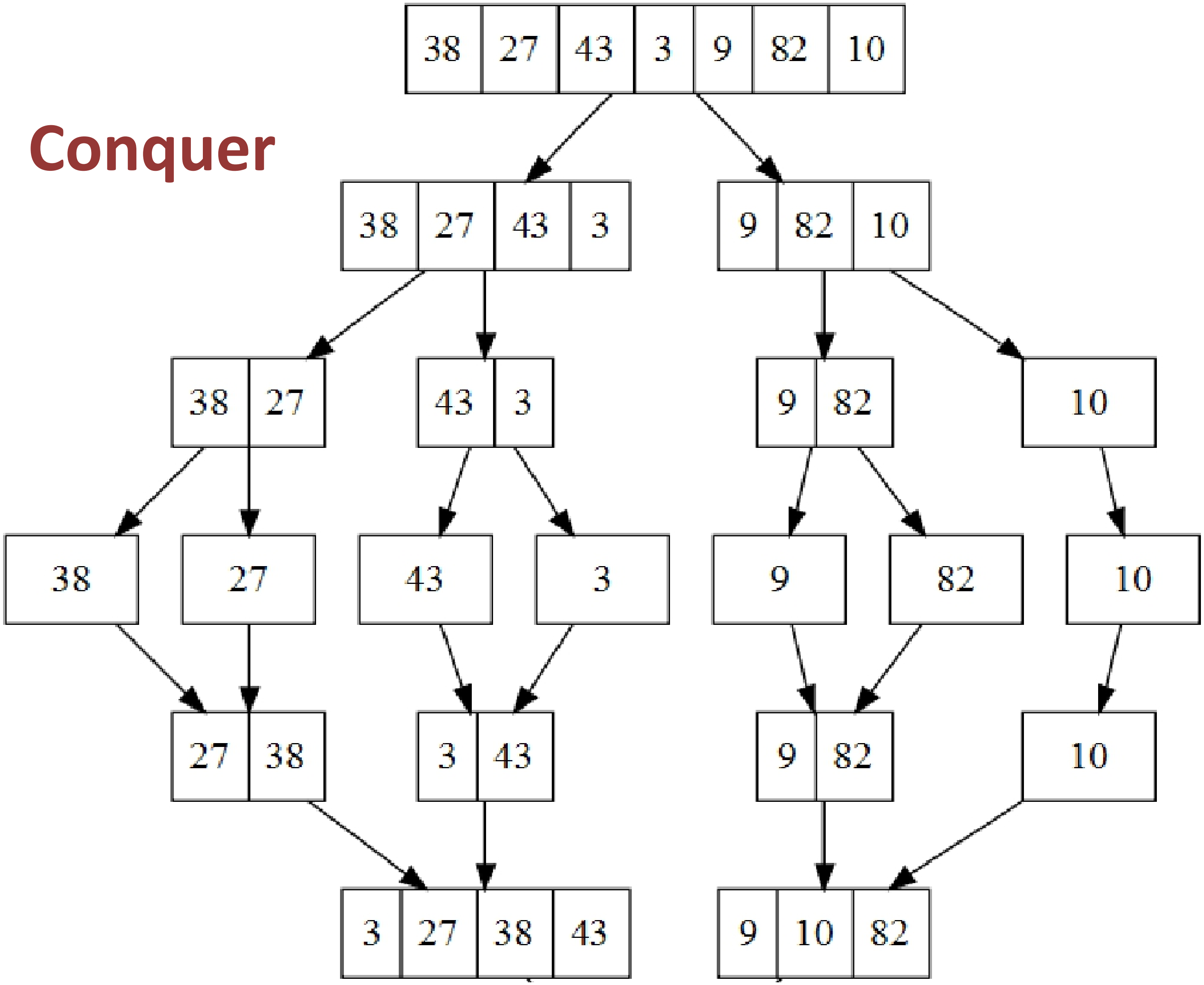
Conquer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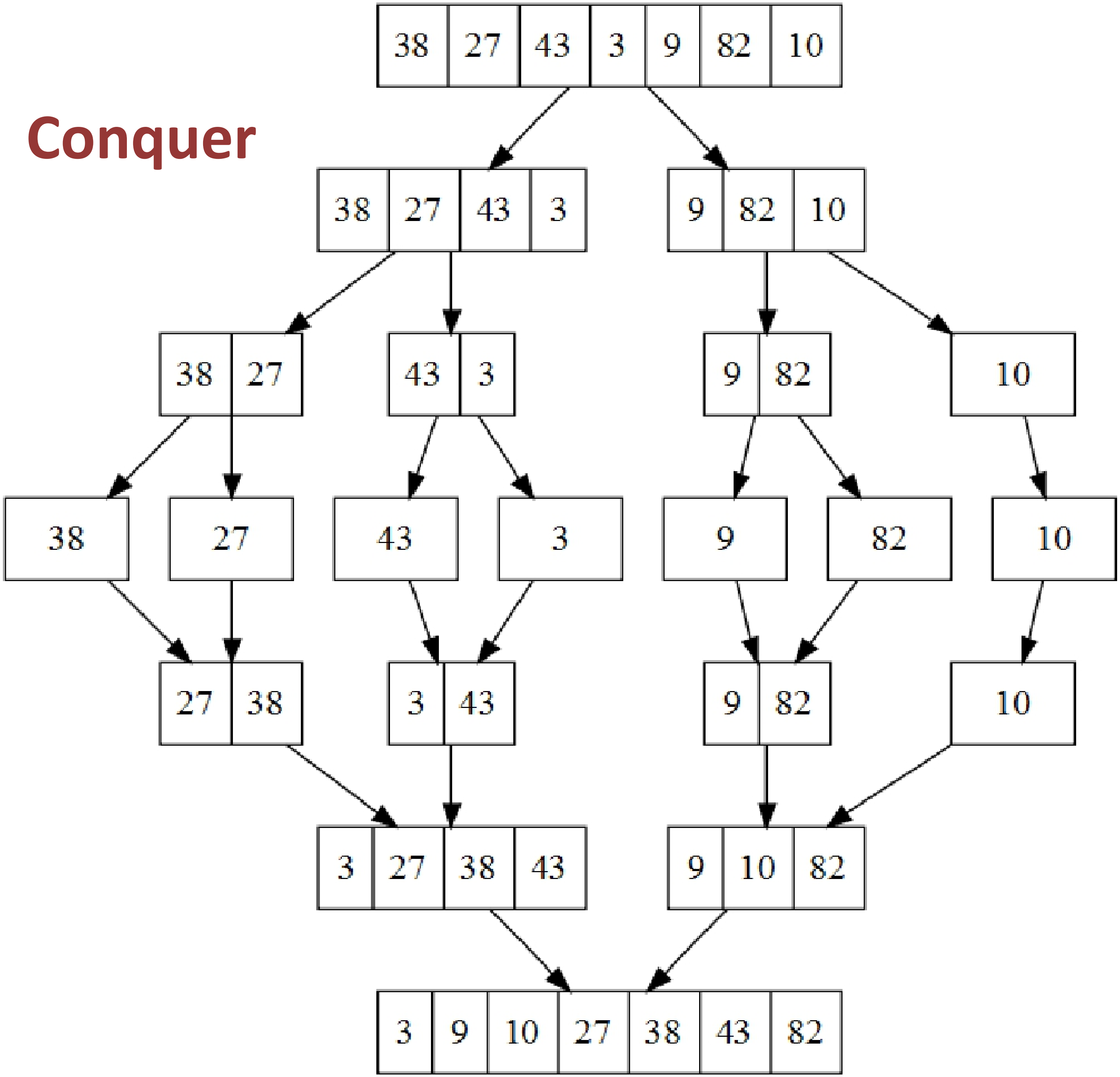
Conquer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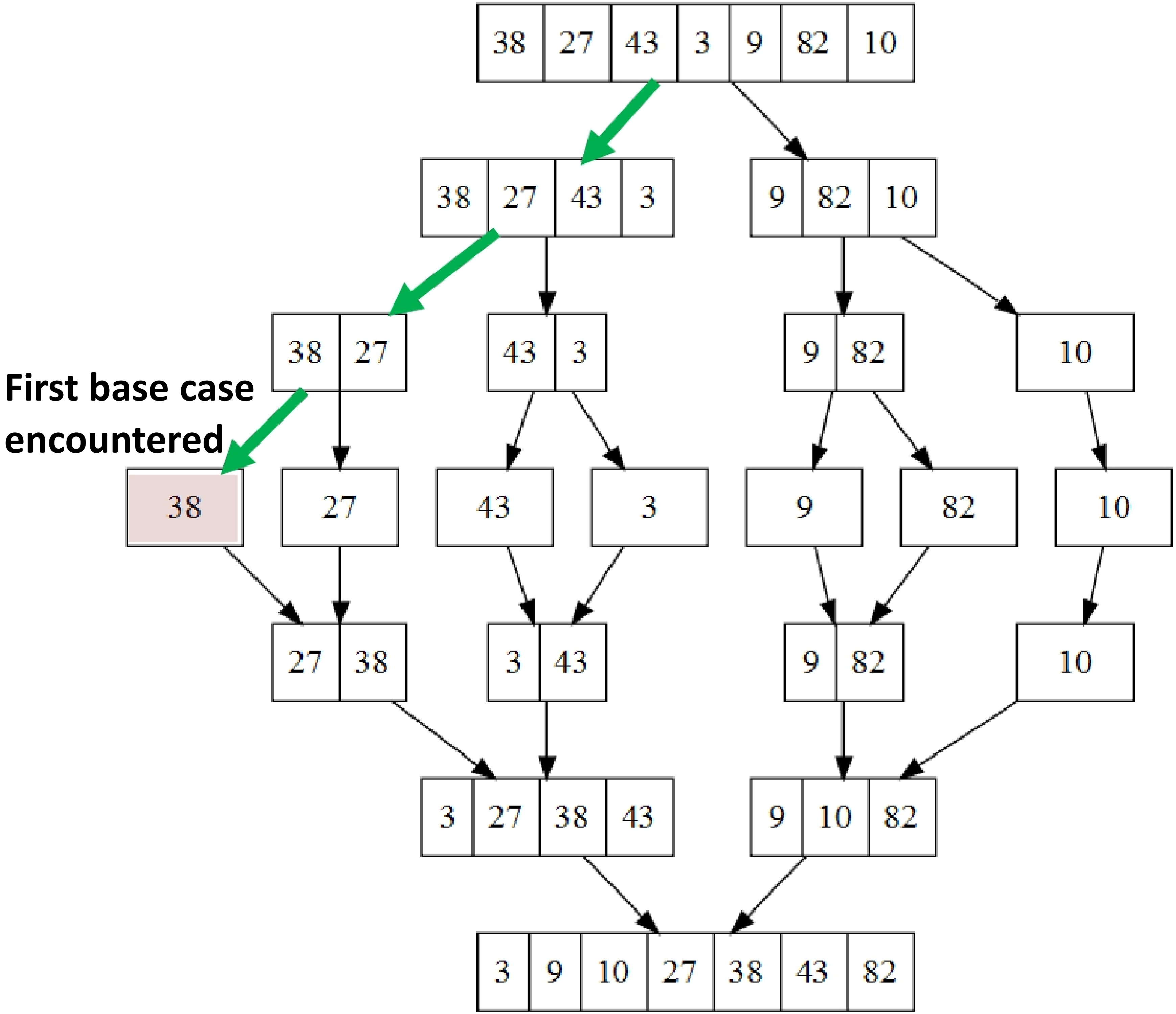
3 27 38 43
9 10 82
3 9 10 27 38 43 82

```
public class Merge
{
   private static void merge(Comparable[] a, Comparable[] aux, int lo, int mid, int hi)
   {  /* as before */  }

   private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
   {
      if (hi <= lo) return;
      int mid = lo + (hi - lo) / 2;
      sort (a, aux, lo, mid);
      sort (a, aux, mid+1, hi);
      merge(a, aux, lo, mid, hi);
   }

   public static void sort(Comparable[] a)
   {
      aux = new Comparable[a.length];
      sort(a, aux, 0, a.length - 1);
   }
}
```



38 27 43 3 9 82 10
38 27 43 3
9 82 10
38 27
43 3
9 82
10
First base case encountered
38
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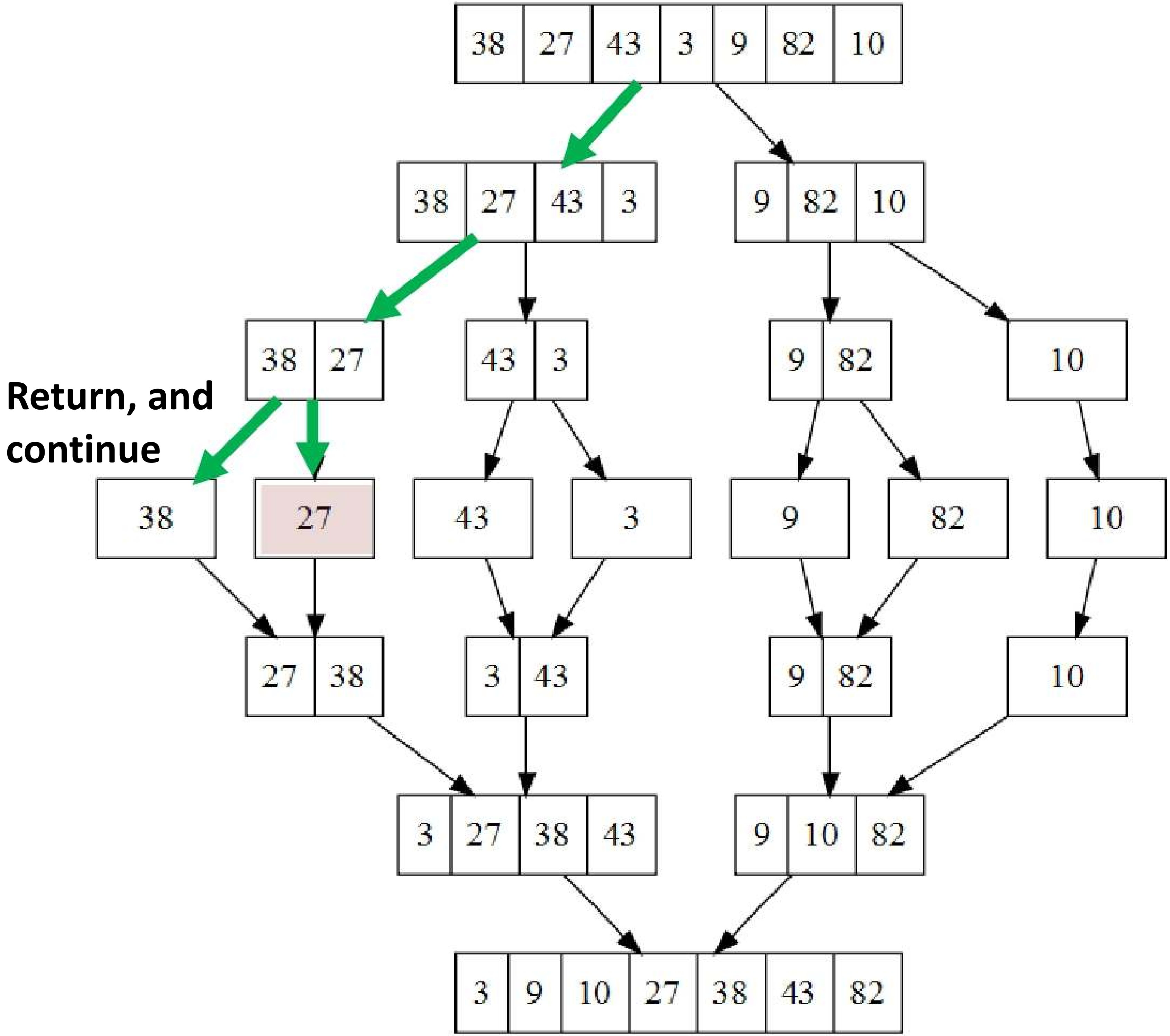
9
82
10
27 38
3 43
9 82
10
3 27 38 43
9 10 82
3 9 10 27 38 43 82



Return, and continue
38 27 43 3 9 82 10
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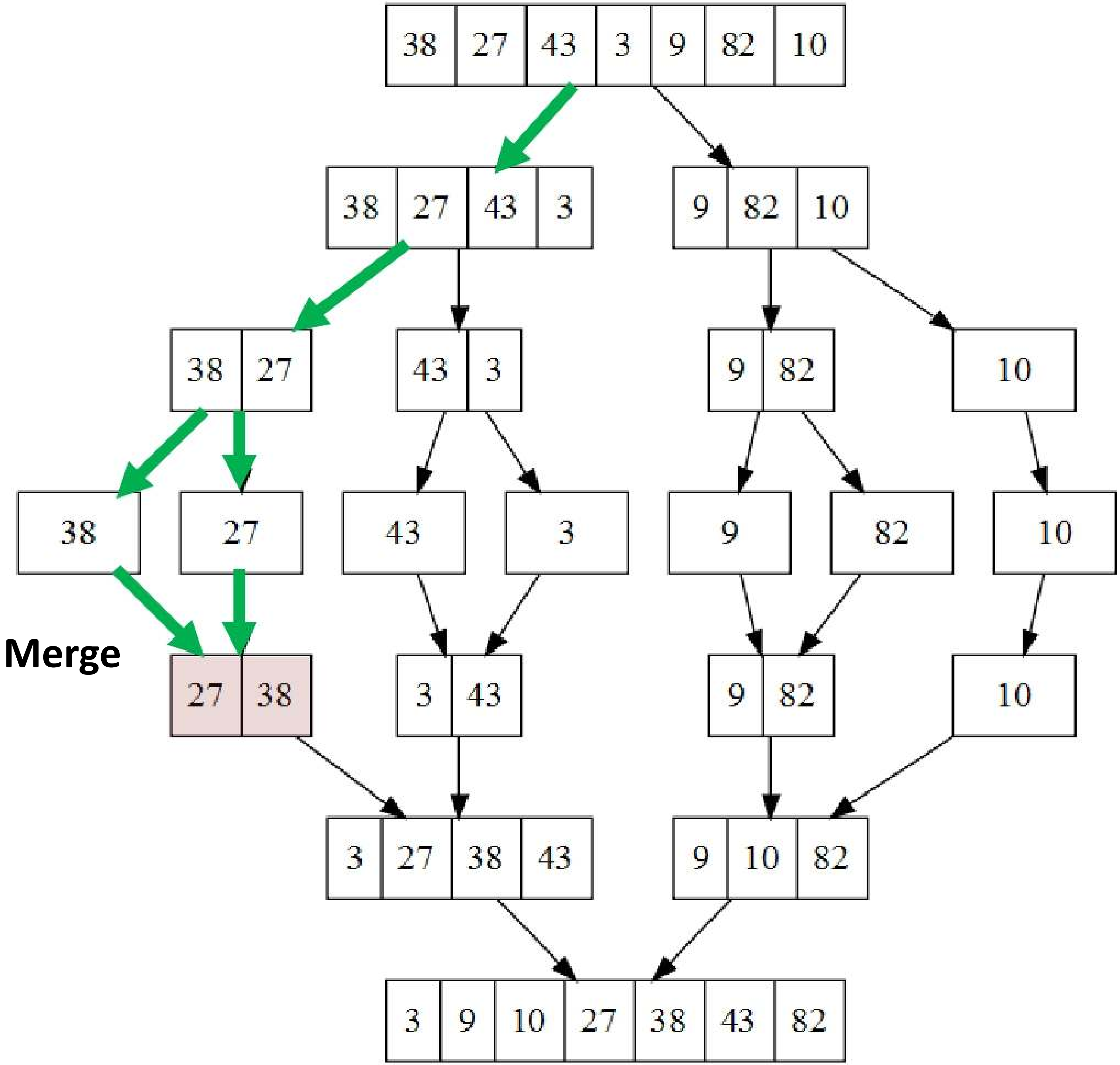
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Merge
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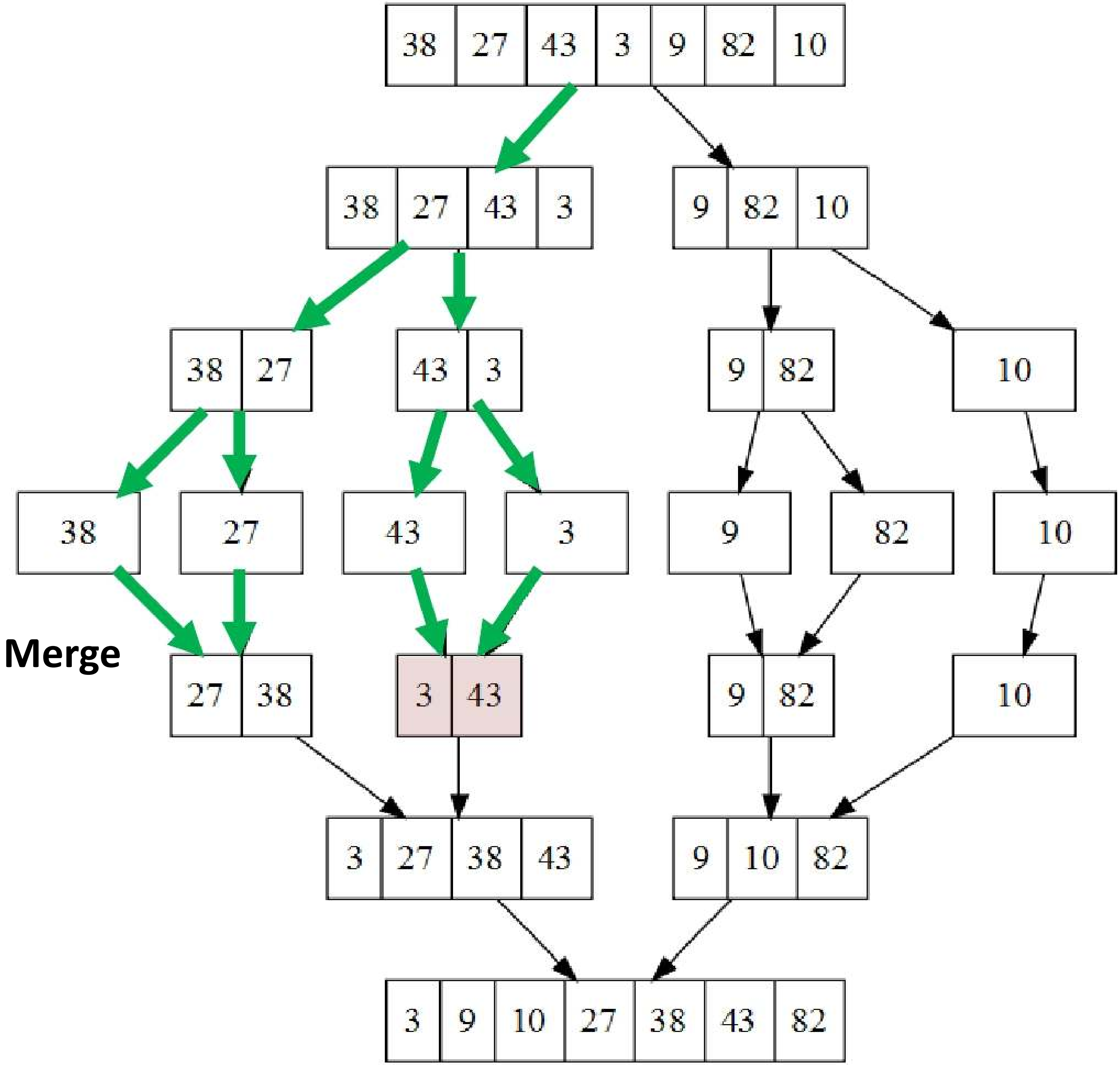
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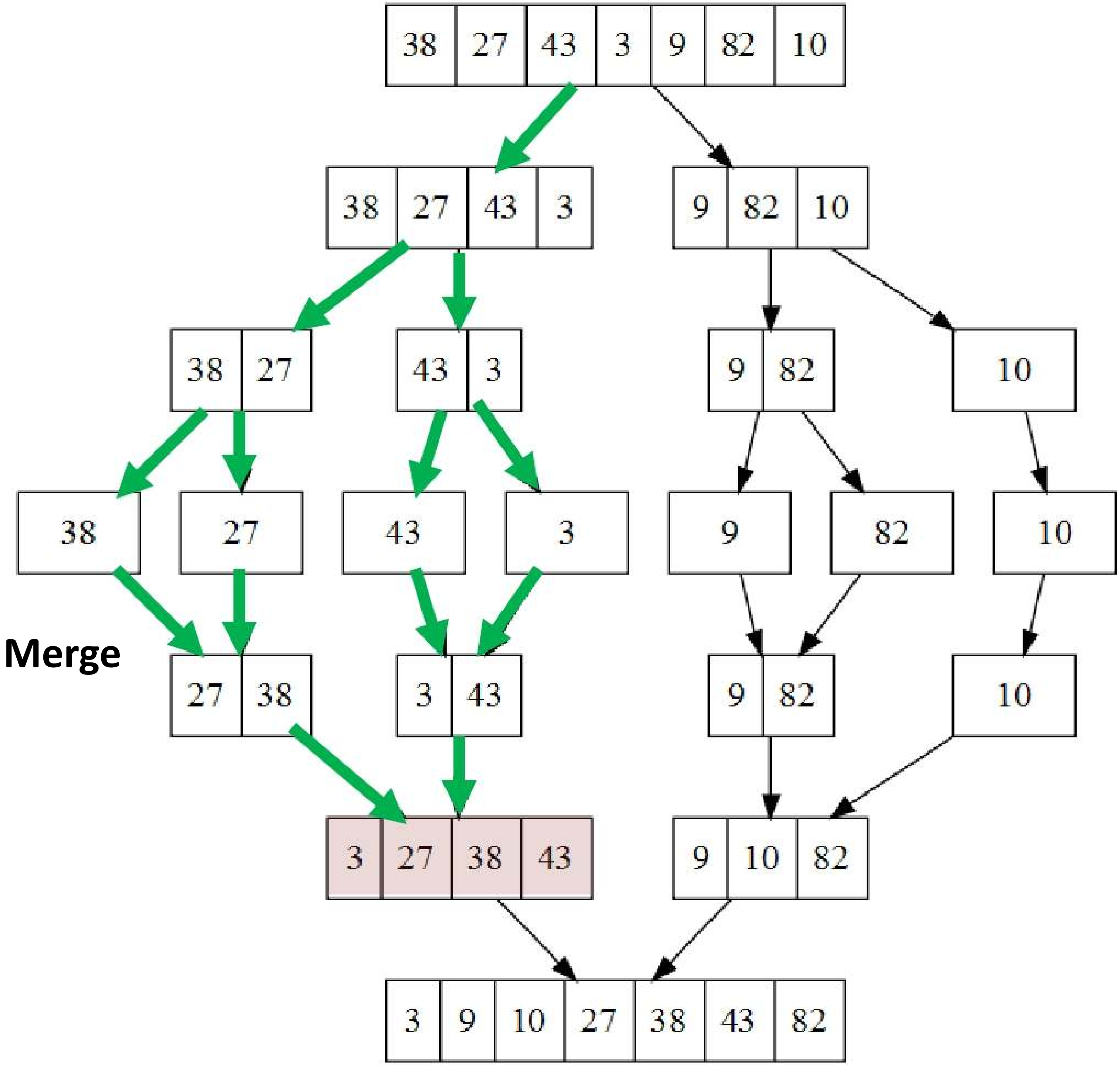

38 27 43 3 9 82 10
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Merge
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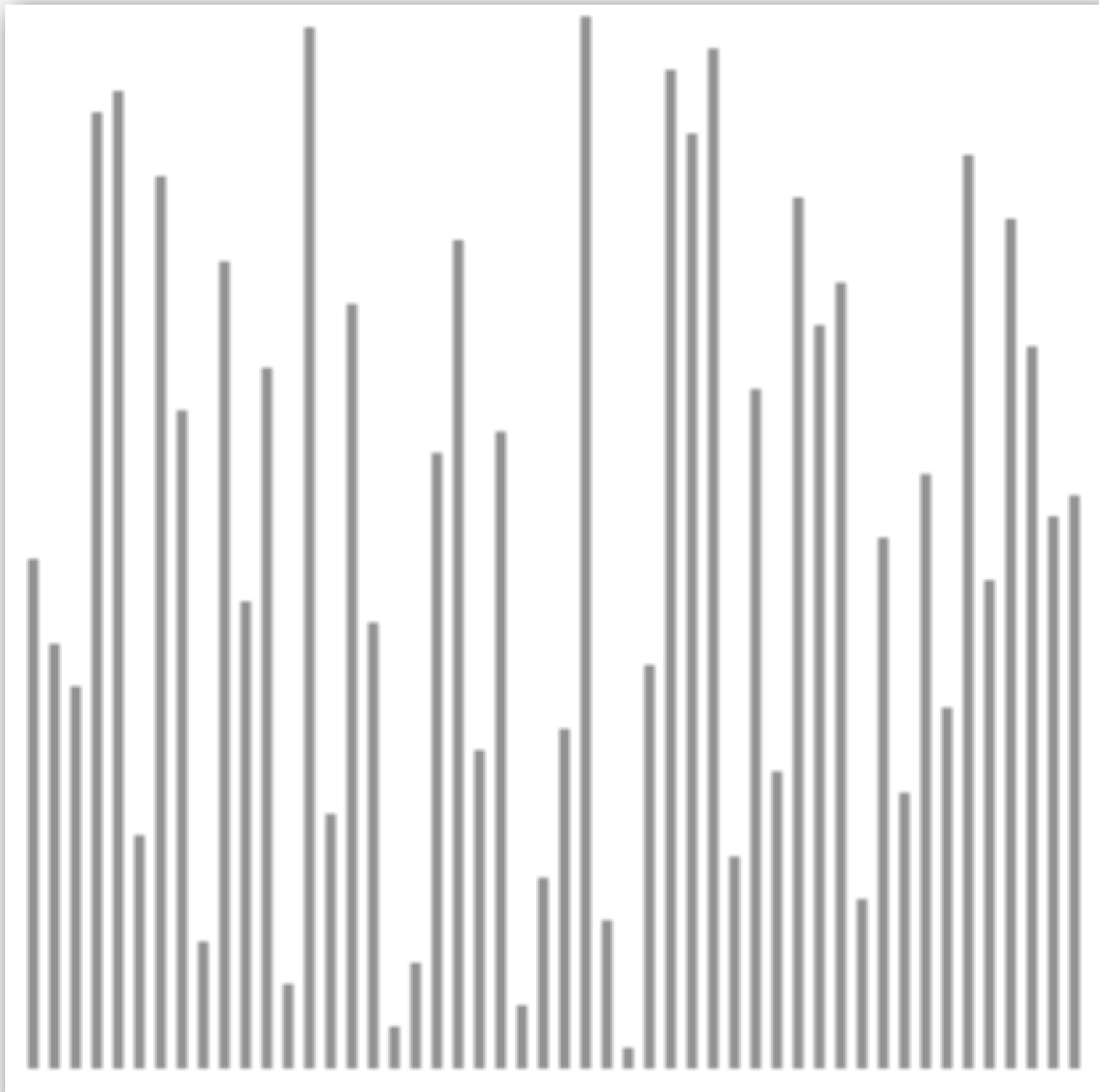
3 27 38 43
9 10 82
3 9 10 27 38 43 82

Mergesort: animation

50 random items



http://www.sorting-algorithms.com/merge-sort

▲ algorithm position
in order
current subarray
not in order

Mergesort: empirical analysis

Running time estimates:

- Laptop executes 10^8 compares/second.
- Supercomputer executes 10^{12} compares/second.

	insertion sort (N^2)			mergesort (N log N)		
computer	thousand	million	billion	thousand	million	billion
home	instant	2.8 hours	317 years	instant	1 second	18 min
super	instant	1 second	1 week	instant	instant	instant

Merge Sort Analysis

Cost Analysis

- What's the cost of mergesort?
- Recurrence relation: T(N) = 2*T(N/2) + N

O(N*logN)

This is called log-linear cost.

Divide-and-conquer recurrence: proof by expansion

Proposition. If $D(N)$ satisfies $D(N) = 2\,D(N/2) + N$ for $N > 1$, with $D(1) = 0$, then $D(N) = N \lg N$.

Pf 2. [assuming N is a power of 2]

$D(N) = 2\,D(N/2) + N$	given
$D(N)\,/\,N = 2\,D(N/2)\,/\,N + 1$	divide both sides by N
$= D(N/2)\,/\,(N/2) + 1$	algebra
$= D(N/4)\,/\,(N/4) + 1 + 1$	apply to first term
$= D(N/8)\,/\,(N/8) + 1 + 1 + 1$	apply to first term again
$\ldots$	
$= D(N/N)\,/\,(N/N) + 1 + 1 + \ldots + 1$	stop applying, D(1) = 0
$= \lg N$	

Merge Sort

Is this a lot better than simple sorting?

# of elements	N^2	N logN
10	100	10
100	10,000	200
1,000	1,000,000	3,000
10,000	100,000,000	40,000
…	…	…

Mergesort: practical improvements

Use insertion sort for small subarrays.

- Mergesort has too much overhead for tiny subarrays.
- Cutoff to insertion sort for ≈ 7 items.

```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
{
   if (hi <= lo + CUTOFF - 1) Insertion.sort(a, lo, hi);
   int mid = lo + (hi - lo) / 2;
   sort (a, aux, lo, mid);
   sort (a, aux, mid+1, hi);
   merge(a, aux, lo, mid, hi);
}
```

Mergesort: practical improvements

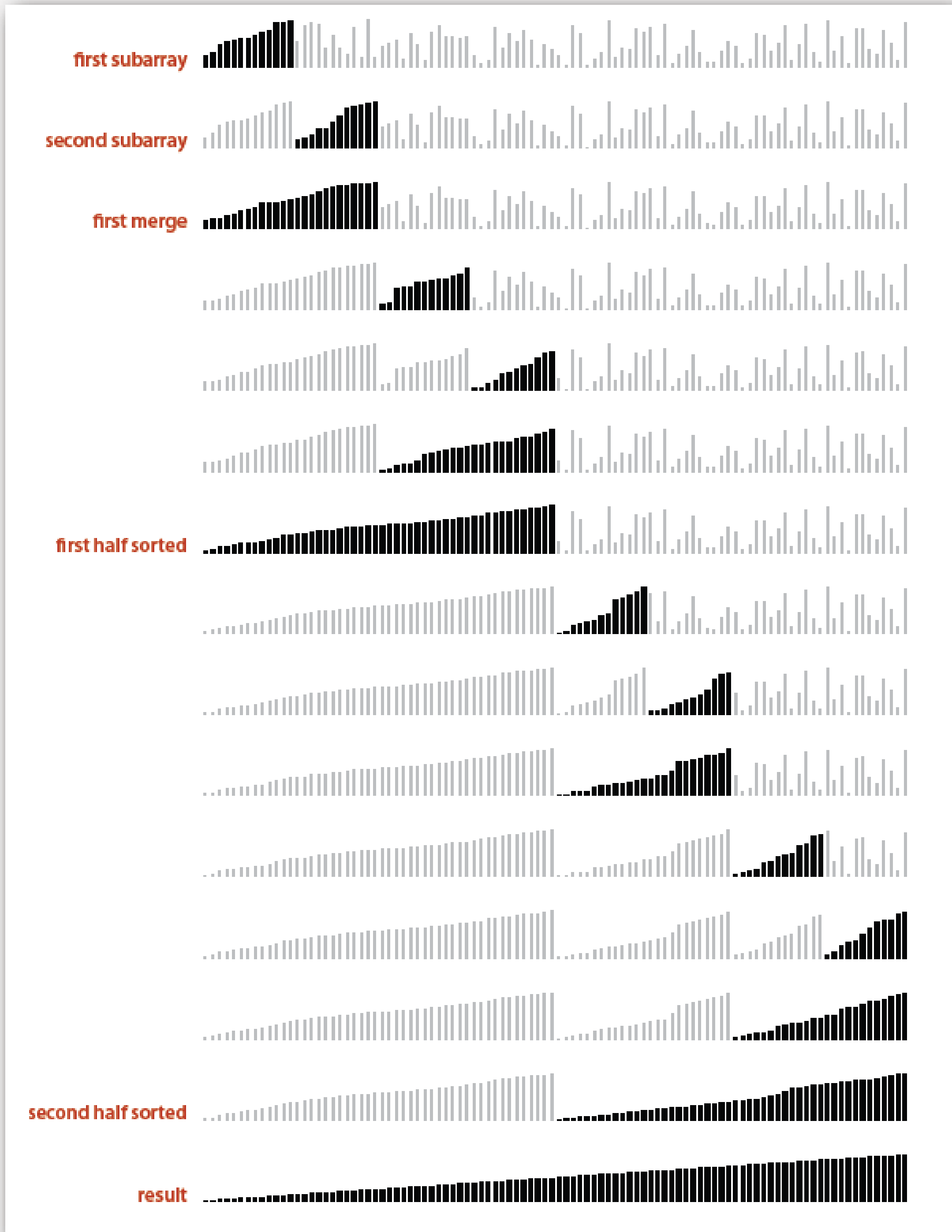
Stop if already sorted.

- Is biggest item in first half ≤ smallest item in second half?
- Helps for partially-ordered arrays.

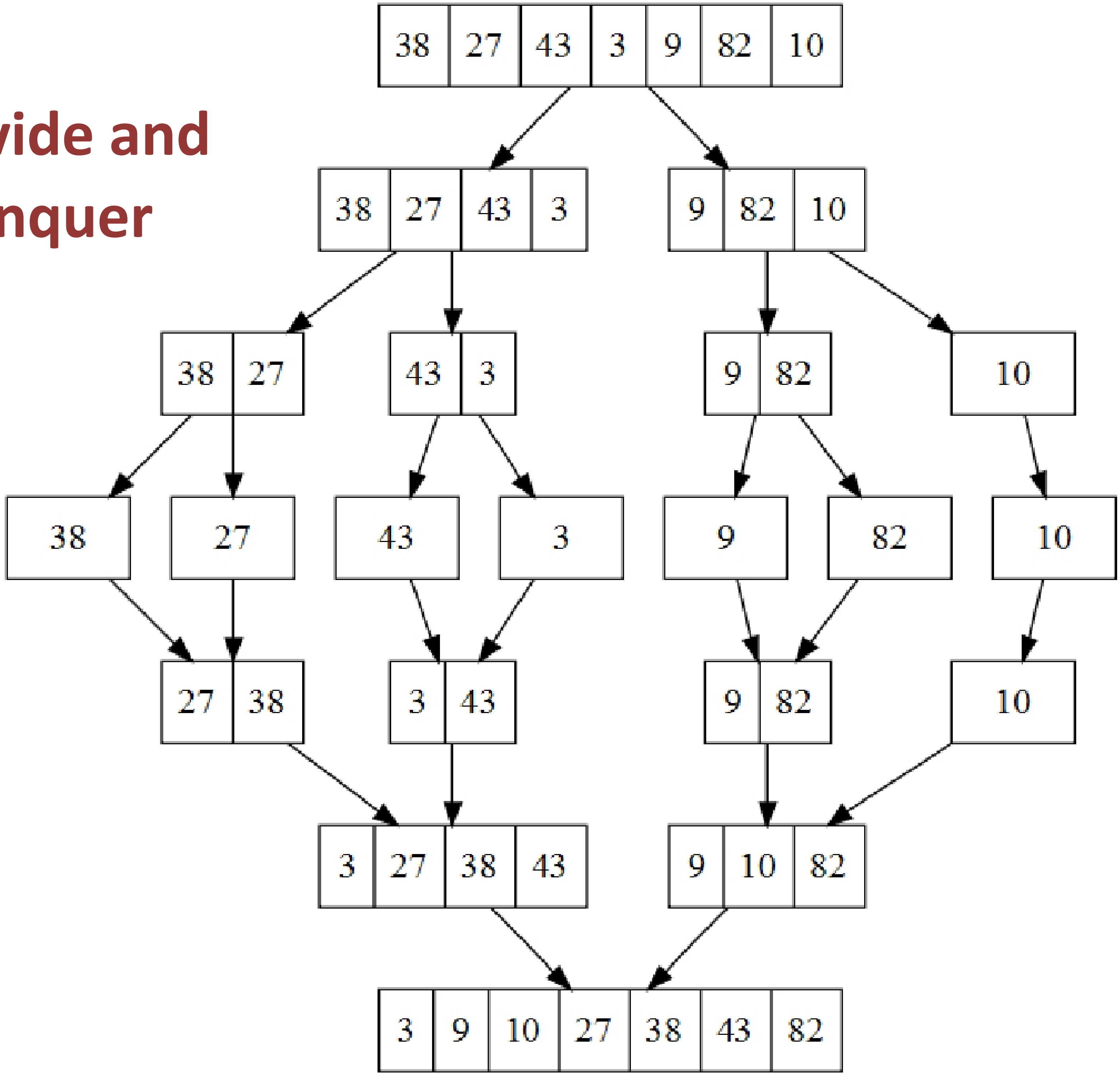
A	B	C	D	E	F	G	H	I	J	M	N	O	P	Q	R	S	T	U	V
A	B	C	D	E	F	G	H	I	J	M	N	O	P	Q	R	S	T	U	V

```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
{
   if (hi <= lo) return;
   int mid = lo + (hi - lo) / 2;
   sort (a, aux, lo, mid);
   sort (a, aux, mid+1, hi);
   if (!less(a[mid+1], a[mid])) return;
   merge(a, aux, lo, mid, hi);
}
```

Mergesort: visualization



Divide and Conquer



Bottom-up MergeSort

1. Every element itself is trivially sorted;
2. Start by merging every two adjacent elements;
3. Then merge every four;
4. Then merge every eight;
5. …
6. Done.

Bottom-up mergesort

Basic plan.

- Pass through array, merging subarrays of size 1.
- Repeat for subarrays of size 2, 4, 8, 16, ...

	a[i]															
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	M	E	R	G	E	S	O	R	T	E	X	A	M	P	L	E
sz = 1																
merge(a, 0, 0, 1)	E	M	R	G	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, 2, 2, 3)	E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, 4, 4, 5)	E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, 6, 6, 7)	E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, 8, 8, 9)	E	M	G	R	E	S	O	R	E	T	X	A	M	P	L	E
merge(a, 10, 10, 11)	E	M	G	R	E	S	O	R	E	T	A	X	M	P	L	E
merge(a, 12, 12, 13)	E	M	G	R	E	S	O	R	E	T	A	X	M	P	L	E
merge(a, 14, 14, 15)	E	M	G	R	E	S	O	R	E	T	A	X	M	P	E	L
sz = 2																
merge(a, 0, 1, 3)	E	G	M	R	E	S	O	R	E	T	A	X	M	P	E	L
merge(a, 4, 5, 7)	E	G	M	R	E	O	R	S	E	T	A	X	M	P	E	L
merge(a, 8, 9, 11)	E	G	M	R	E	O	R	S	A	E	T	X	M	P	E	L
merge(a, 12, 13, 15)	E	G	M	R	E	O	R	S	A	E	T	X	E	L	M	P
sz = 4																
merge(a, 0, 3, 7)	E	E	G	M	O	R	R	S	A	E	T	X	E	L	M	P
merge(a, 8, 11, 15)	E	E	G	M	O	R	R	S	A	E	E	L	M	P	T	X
sz = 8																
merge(a, 0, 7, 15)	A	E	E	E	E	G	L	M	M	O	P	R	R	S	T	X

Bottom line. No recursion needed!

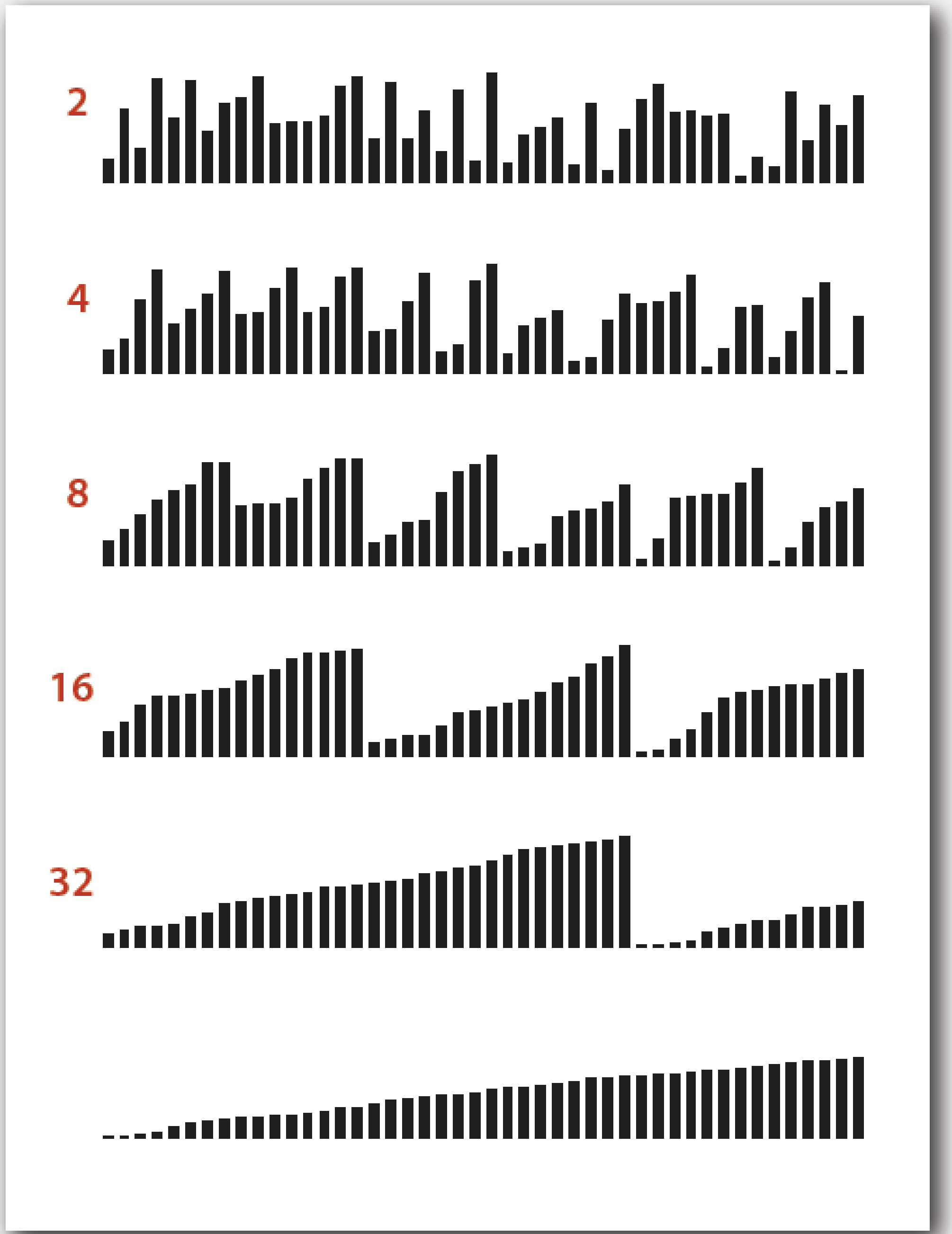
Bottom-up mergesort: Java implementation

```
public class MergeBU
{
   private static Comparable[] aux;

   private static void merge(Comparable[] a, int lo, int mid, int hi)
   {  /* as before */  }

   public static void sort(Comparable[] a)
   {
      int N = a.length;
      aux = new Comparable[N];
      for (int sz = 1; sz < N; sz = sz+sz)
         for (int lo = 0; lo < N-sz; lo += sz+sz)
            merge(a, lo, lo+sz-1, Math.min(lo+sz+sz-1, N-1));
   }
}
```

Bottom-up mergesort: visual trace



Summary

- Merging two sorted array is a key step in merge sort.
- Merge sort uses a divide and conquer approach.
- It repeatedly splits an input array to two sub-arrays, sort each sub-array, and merge the two.
- It requires O(N*logN) time.

- On the downside, it requires additional memory space (the workspace array).