# 1.5 UNION FIND


- dynamic connectivity
- quick find
- quick union
- improvements
- applications

## Subtext of today's lecture (and this course)

Steps to developing a usable algorithm.

- Model the problem.
- Find an algorithm to solve it.
- Fast enough? Fits in memory?
- If not, figure out why.
- Find a way to address the problem.
- Iterate until satisfied.

The scientific method.

Mathematical analysis.

- dynamic connectivity
- quick find
- quick union
- improvements
- applications

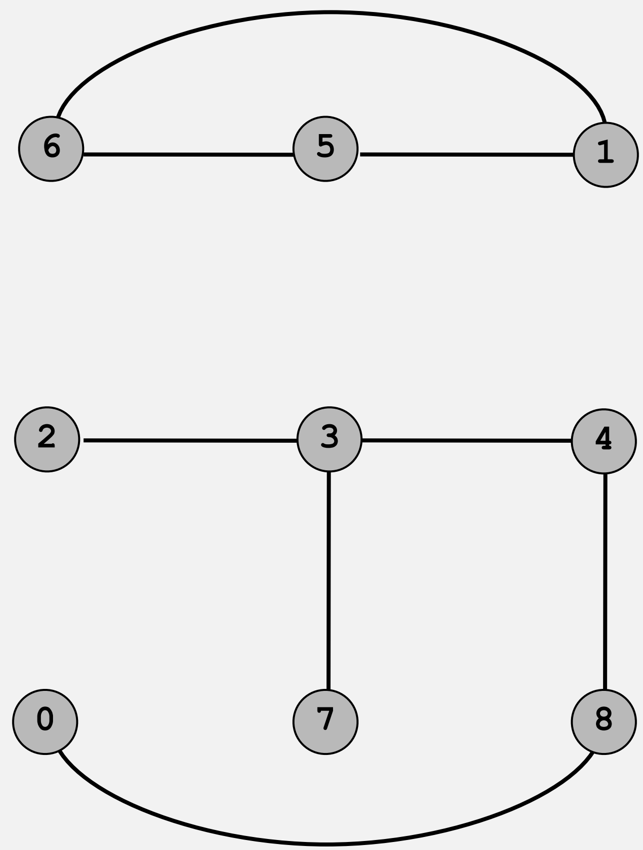
# Dynamic connectivity

## Given a set of objects

more difficult problem: find the path

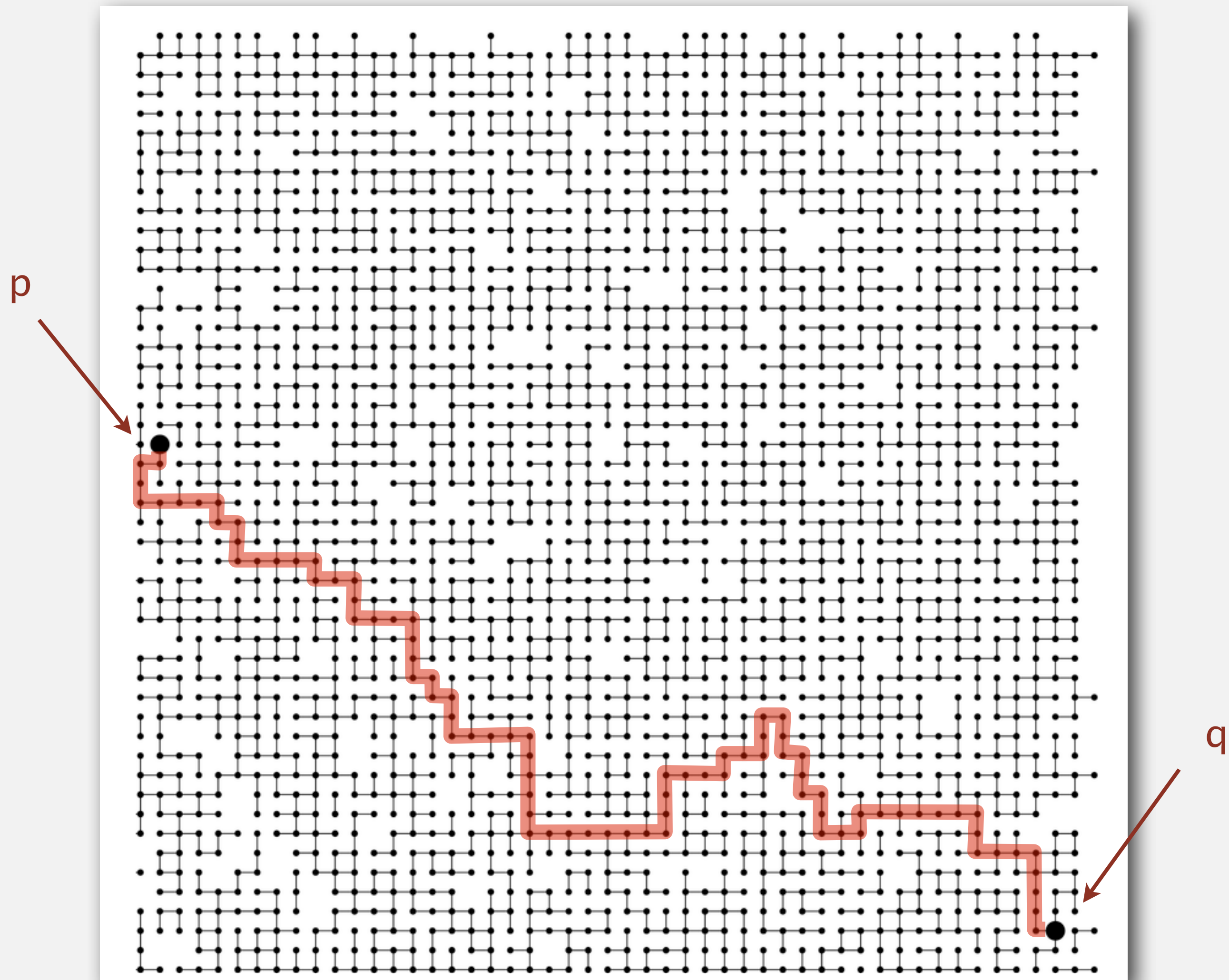
- **Union:** connect two objects.
- **Connected:** is there a path connecting the two objects?

```
union(3, 4)
union(8, 0)
union(2, 3)
union(5, 6)
connected(0, 2)   no
connected(2, 4)   yes
union(5, 1)
union(7, 3)
union(1, 6)
union(4, 8)
connected(0, 2)   yes
connected(2, 4)   yes
```



## Connectivity example

**Q.** Is there a path from p to q?




**A.** Yes.

# Modeling the objects

Dynamic connectivity applications involve manipulating objects of all types.

- Pixels in a digital photo.
- Computers in a network.
- Variable names in Fortran.
- Friends in a social network.
- Transistors in a computer chip.
- Elements in a mathematical set.
- Metallic sites in a composite system.

When programming, convenient to name sites 0 to N-1.

- Use integers as array index.
- Suppress details not relevant to union-find.

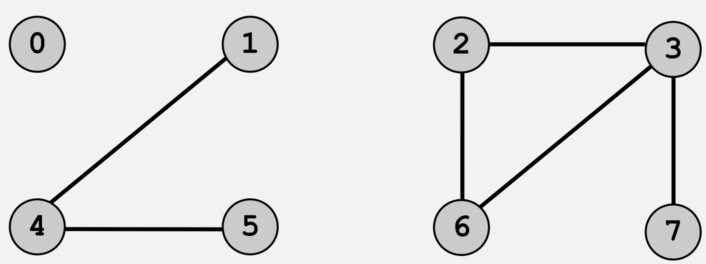
can use symbol table to translate from site names to integers: stay tuned (Chapter 3)

# Modeling the connections

We assume "is connected to" is an equivalence relation:

- Reflexive: $p$ is connected to $p$.
- Symmetric: if $p$ is connected to $q$, then $q$ is connected to $p$.
- Transitive: if $p$ is connected to $q$ and $q$ is connected to $r$, then $p$ is connected to $r$.

Connected components. Maximal set of objects that are mutually connected.



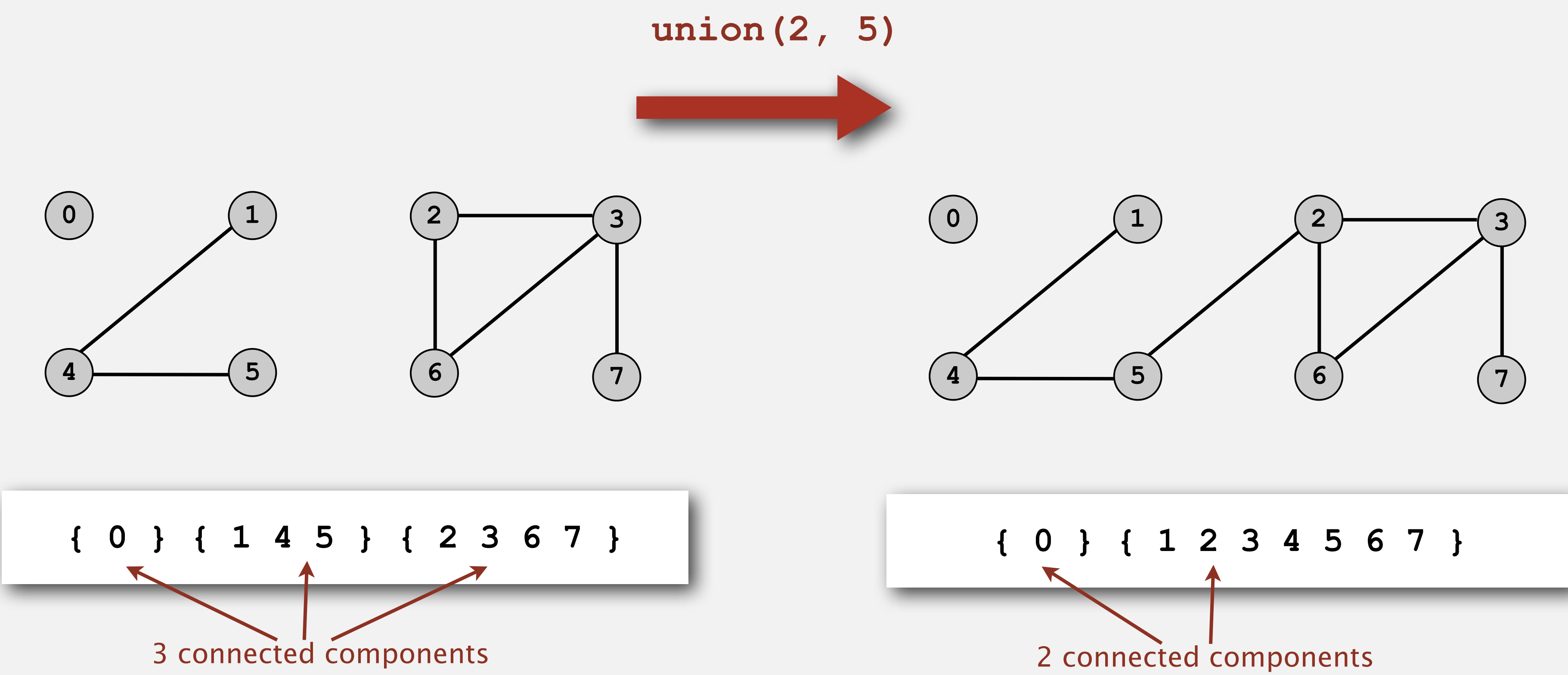

{ 0 } { 1 4 5 } { 2 3 6 7 }

3 connected components

# Implementing the operations

**Find query.** Check if two objects are in the same component.

**Union command.** Replace components containing two objects with their union.



## Union-find data type (API)

**Goal.** Design efficient data structure for union-find.

- Number of objects $N$ can be huge.
- Number of operations $M$ can be huge.
- Find queries and union commands may be intermixed.

| public class **UF** | | |
|---|---|---|
| | `UF(int N)` | *initialize union-find data structure with N objects (0 to N-1)* |
| `void` | `union(int p, int q)` | *add connection between p and q* |
| `boolean` | `connected(int p, int q)` | *are p and q in the same component?* |
| `int` | `find(int p)` | *component identifier for p (0 to N-1)* |
| `int` | `count()` | *number of components* |

# Dynamic-connectivity client

- Read in number of objects $N$ from standard input.
- Repeat:
  - read in pair of integers from standard input
  - write out pair if they are not already connected

```
public static void main(String[] args)
{
   int N = StdIn.readInt();
   UF uf = new UF(N);
   while (!StdIn.isEmpty())
   {
      int p = StdIn.readInt();
      int q = StdIn.readInt();
      if (uf.connected(p, q)) continue;
      uf.union(p, q);
      StdOut.println(p + " " + q);
   }
}
```

```
% more tiny.txt
10
4 3
3 8
6 5
9 4
2 1
8 9
5 0
7 2
6 1
1 0
6 7
```

- dynamic connectivity
- **quick find**
- quick union
- improvements
- applications

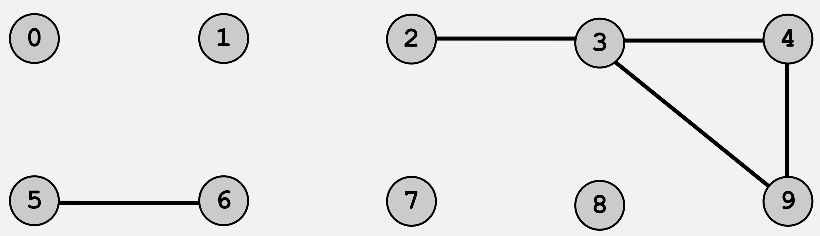
# Quick-find [eager approach]

## Data structure.

- Integer array `id[]` of size `N`.
- Interpretation: `p` and `q` are connected iff they have the same id.

| i | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| id[i] | 0 | 1 | 9 | 9 | 9 | 6 | 6 | 7 | 8 | 9 |

5 and 6 are connected
2, 3, 4, and 9 are connected



# Quick-find [eager approach]

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| id[i] | 0 | 1 | 9 | 9 | 9 | 6 | 6 | 7 | 8 | 9 |

**Find.** Check if `p` and `q` have the same id.

`id[3] = 9`; `id[6] = 6`
3 and 6 are not connected

# Quick-find [eager approach]

**Data structure.**

- Integer array `id[]` of size `N`.
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| id[i] | 0 | 1 | 9 | 9 | 9 | 6 | 6 | 7 | 8 | 9 |

**Find.** Check if `p` and `q` have the same id.

`id[3] = 9; id[6] = 6`
3 and 6 are not connected

**Union.** To merge components containing `p` and `q`, change all entries whose `id[]` equals `id[p]` to `id[q]`.

| i | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| id[i] | 0 | 1 | 6 | 6 | 6 | 6 | 6 | 7 | 8 | 6 |

after union of 3 and 6

problem: many values can change

# Quick-find example

| p | q | id[] 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 3 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| | | 0 | 1 | 2 | 3 | 3 | 5 | 6 | 7 | 8 | 9 |
| 3 | 8 | 0 | 1 | 2 | 3 | 3 | 5 | 6 | 7 | 8 | 9 |
| | | 0 | 1 | 2 | 8 | 8 | 5 | 6 | 7 | 8 | 9 |
| 6 | 5 | 0 | 1 | 2 | 8 | 8 | 5 | 6 | 7 | 8 | 9 |
| | | 0 | 1 | 2 | 8 | 8 | 5 | 5 | 7 | 8 | 9 |
| 9 | 4 | 0 | 1 | 2 | 8 | 8 | 5 | 5 | 7 | 8 | 9 |
| | | 0 | 1 | 2 | 8 | 8 | 5 | 5 | 7 | 8 | 8 |
| 2 | 1 | 0 | 1 | 2 | 8 | 8 | 5 | 5 | 7 | 8 | 8 |
| | | 0 | 1 | 1 | 8 | 8 | 5 | 5 | 7 | 8 | 8 |
| 8 | 9 | 0 | 1 | 1 | 8 | 8 | 5 | 5 | 7 | 8 | 8 |
| 5 | 0 | 0 | 1 | 1 | 8 | 8 | 5 | 5 | 7 | 8 | 8 |
| | | 0 | 1 | 1 | 8 | 8 | 0 | 0 | 7 | 8 | 8 |
| 7 | 2 | 0 | 1 | 1 | 8 | 8 | 0 | 0 | 7 | 8 | 8 |
| | | 0 | 1 | 1 | 8 | 8 | 0 | 0 | 1 | 8 | 8 |
| 6 | 1 | 0 | 1 | 1 | 8 | 8 | 0 | 0 | 1 | 8 | 8 |
| | | 1 | 1 | 1 | 8 | 8 | 1 | 1 | 1 | 8 | 8 |
| 1 | 0 | 1 | 1 | 1 | 8 | 8 | 1 | 1 | 1 | 8 | 8 |
| 6 | 7 | 1 | 1 | 1 | 8 | 8 | 1 | 1 | 1 | 8 | 8 |

`id[p]` *and* `id[q]` *differ, so* `union()` *changes entries equal to* `id[p]` *to* `id[q]` *(in red)*

`id[p]` *and* `id[q]` *match, so no change*

# Quick-find: Java implementation

```
public class QuickFindUF
{
   private int[] id;

   public QuickFindUF(int N)
   {
      id = new int[N];
      for (int i = 0; i < N; i++)
         id[i] = i;
   }

   public boolean connected(int p, int q)
   {  return id[p] == id[q];  }

   public void union(int p, int q)
   {
      int pid = id[p];
      int qid = id[q];
      for (int i = 0; i < id.length; i++)
         if (id[i] == pid) id[i] = qid;
   }
}
```

set id of each object to itself
(N array accesses)

check whether `p` and `q`
are in the same component
(2 array accesses)

change all entries with `id[p]` to `id[q]`
(linear number of array accesses)

# Quick-find is too slow

**Cost model.** Number of array accesses (for read or write).

| algorithm | init | union | connected |
|---|---|---|---|
| quick-find | N | N | 1 |

**order of growth of number of array accesses**

**Quick-find defect.**

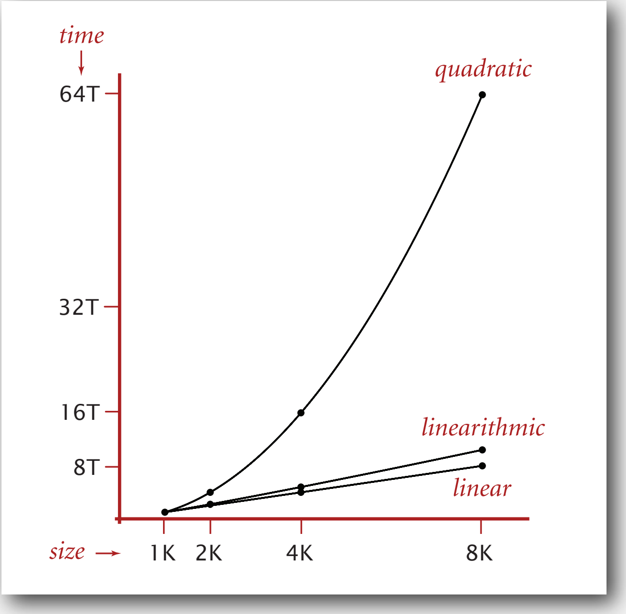
- Union too expensive.
- Trees are flat, but too expensive to keep them flat.
- Ex. Takes $N^2$ array accesses to process sequence of $N$ union commands on $N$ objects.

# Quadratic algorithms do not scale

## Rough standard (for now).

- $10^9$ operations per second.
- $10^9$ words of main memory.
- Touch all words in approximately 1 second.

a truism (roughly) since 1950!




## Ex. Huge problem for quick-find.

- $10^9$ union commands on $10^9$ objects.
- Quick-find takes more than $10^{18}$ operations.
- 30+ years of computer time!

## Paradoxically, quadratic algorithms get worse with newer equipment.

- New computer may be 10x as fast.
- But, has 10x as much memory so problem may be 10x bigger.
- With quadratic algorithm, takes 10x as long!

‣ dynamic connectivity
‣ quick find
‣ **quick union**
‣ improvements
‣ applications

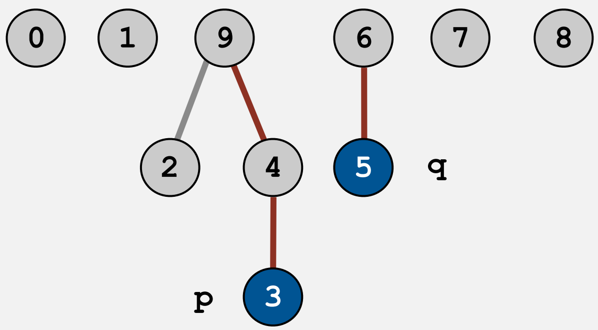
## Quick-union [lazy approach]

### Data structure.

- Integer array `id[]` of size `N`.
- Interpretation: `id[i]` is parent of `i`.
- Root of `i` is `id[id[id[...id[i]...]]]`.

keep going until it doesn't change

| i | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| id[i] | 0 | 1 | 9 | 4 | 9 | 6 | 6 | 7 | 8 | 9 |



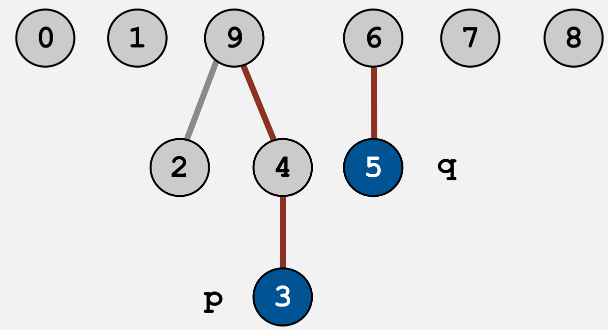

3's root is 9; 5's root is 6

## Data structure.

- Integer array `id[]` of size `N`.
- Interpretation: `id[i]` is parent of `i`.
- Root of `i` is `id[id[id[...id[i]...]]]`.

keep going until it doesn't change

| i | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| id[i] | 0 | 1 | 9 | 4 | 9 | 6 | 6 | 7 | 8 | 9 |




3's root is 9; 5's root is 6
3 and 5 are not connected

**Find.** Check if `p` and `q` have the same root.

# Quick-union [lazy approach]

## Data structure.

- Integer array `id[]` of size `N`.
- Interpretation: `id[i]` is parent of `i`.
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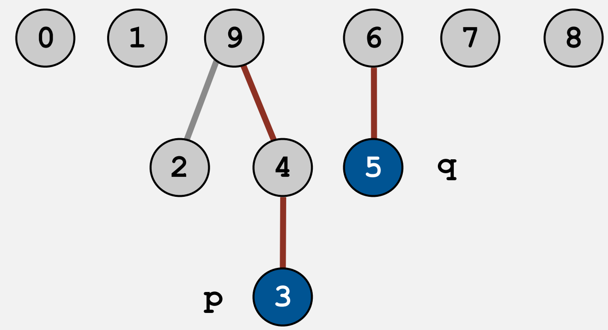
keep going until it doesn't change

| i | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| id[i] | 0 | 1 | 9 | 4 | 9 | 6 | 6 | 7 | 8 | 9 |



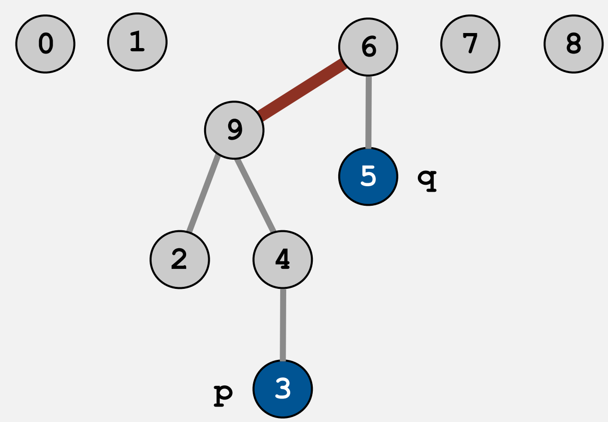

3's root is 9; 5's root is 6
3 and 5 are not connected

**Find.** Check if `p` and `q` have the same root.

**Union.** To merge components containing `p` and `q`, set the id of `p`'s root to the id of `q`'s root.

| i | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| id[i] | 0 | 1 | 9 | 4 | 9 | 6 | 6 | 7 | 8 | 6 |

only one value changes

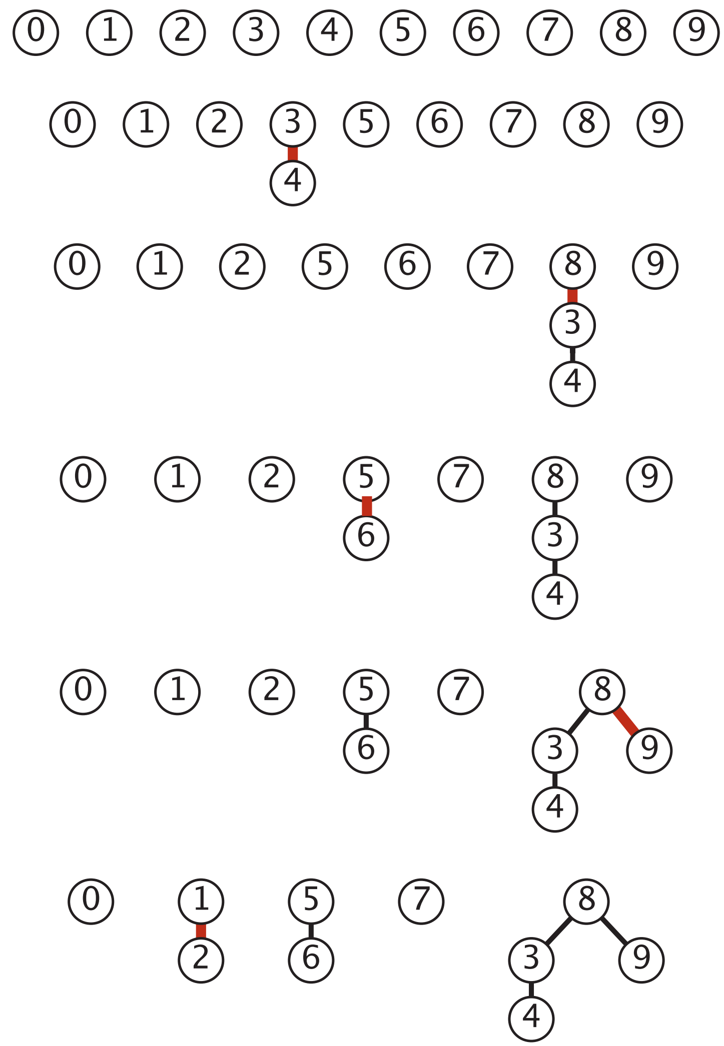


# Quick-union demo

# Quick-union example

id[]

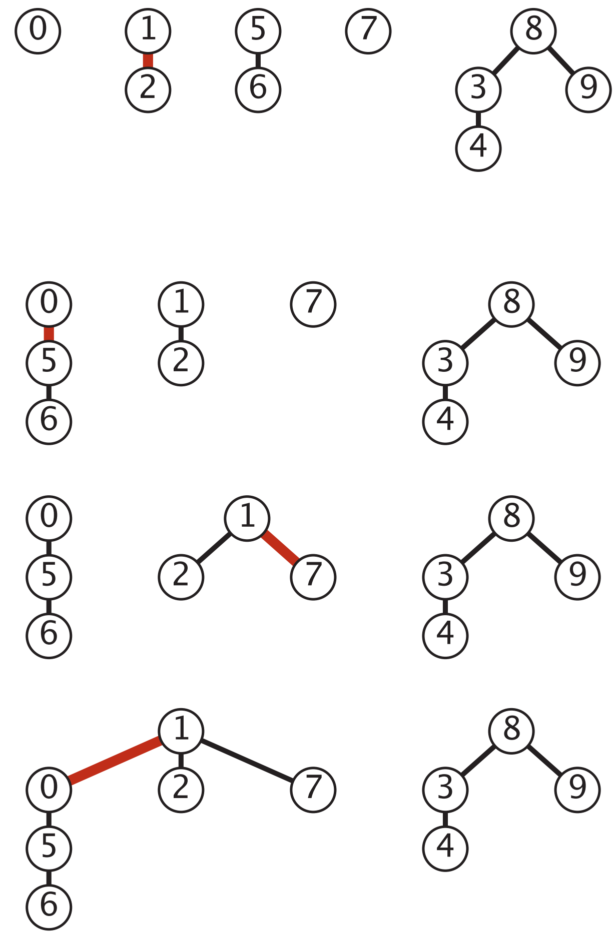
| p | q | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 3 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| | | 0 | 1 | 2 | 3 | 3 | 5 | 6 | 7 | 8 | 9 |
| 3 | 8 | 0 | 1 | 2 | 3 | 3 | 5 | 6 | 7 | 8 | 9 |
| | | 0 | 1 | 2 | 8 | 3 | 5 | 6 | 7 | 8 | 9 |
| 6 | 5 | 0 | 1 | 2 | 8 | 3 | 5 | 6 | 7 | 8 | 9 |
| | | 0 | 1 | 2 | 8 | 3 | 5 | 5 | 7 | 8 | 9 |
| 9 | 4 | 0 | 1 | 2 | 8 | 3 | 5 | 5 | 7 | 8 | 9 |
| | | 0 | 1 | 2 | 8 | 3 | 5 | 5 | 7 | 8 | 8 |
| 2 | 1 | 0 | 1 | 2 | 8 | 3 | 5 | 5 | 7 | 8 | 8 |
| | | 0 | 1 | 1 | 8 | 3 | 5 | 5 | 7 | 8 | 8 |



# Quick-union example

id[]

| p | q | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 8 | 9 | 0 | 1 | 1 | 8 | 3 | 5 | 5 | 7 | 8 | 8 |
| 5 | 0 | 0 | 1 | 1 | 8 | 3 | 5 | 5 | 7 | 8 | 8 |
| | | 0 | 1 | 1 | 8 | 3 | 0 | 5 | 7 | 8 | 8 |
| 7 | 2 | 0 | 1 | 1 | 8 | 3 | 0 | 5 | 7 | 8 | 8 |
| | | 0 | 1 | 1 | 8 | 3 | 0 | 5 | 1 | 8 | 8 |
| 6 | 1 | 0 | 1 | 1 | 8 | 3 | 0 | 5 | 1 | 8 | 8 |
| | | 1 | 1 | 1 | 8 | 3 | 0 | 5 | 1 | 8 | 8 |
| 1 | 0 | 1 | 1 | 1 | 8 | 3 | 0 | 5 | 1 | 8 | 8 |
| 6 | 7 | 1 | 1 | 1 | 8 | 3 | 0 | 5 | 1 | 8 | 8 |



# Quick-union: Java implementation

```
public class QuickUnionUF
{
   private int[] id;

   public QuickUnionUF(int N)
   {
      id = new int[N];
      for (int i = 0; i < N; i++) id[i] = i;
   }

   private int root(int i)
   {
      while (i != id[i]) i = id[i];
      return i;
   }

   public boolean connected(int p, int q)
   {
      return root(p) == root(q);
   }

   public void union(int p, int q)
   {
      int i = root(p);
      int j = root(q);
      id[i] = j;
   }
}
```

set id of each object to itself
(N array accesses)

chase parent pointers until reach root
(depth of i array accesses)

check if p and q have same root
(depth of p and q array accesses)

change root of p to point to root of q
(depth of p and q array accesses)

# Quick-union is also too slow

Cost model. Number of array accesses (for read or write).

| algorithm | init | union | connected |
|---|---|---|---|
| quick-find | N | N | 1 |
| quick-union | N | N † | N |

← worst case

† includes cost of finding roots

## Quick-find defect.

- Union too expensive ($N$ array accesses).
- Trees are flat, but too expensive to keep them flat.

## Quick-union defect.

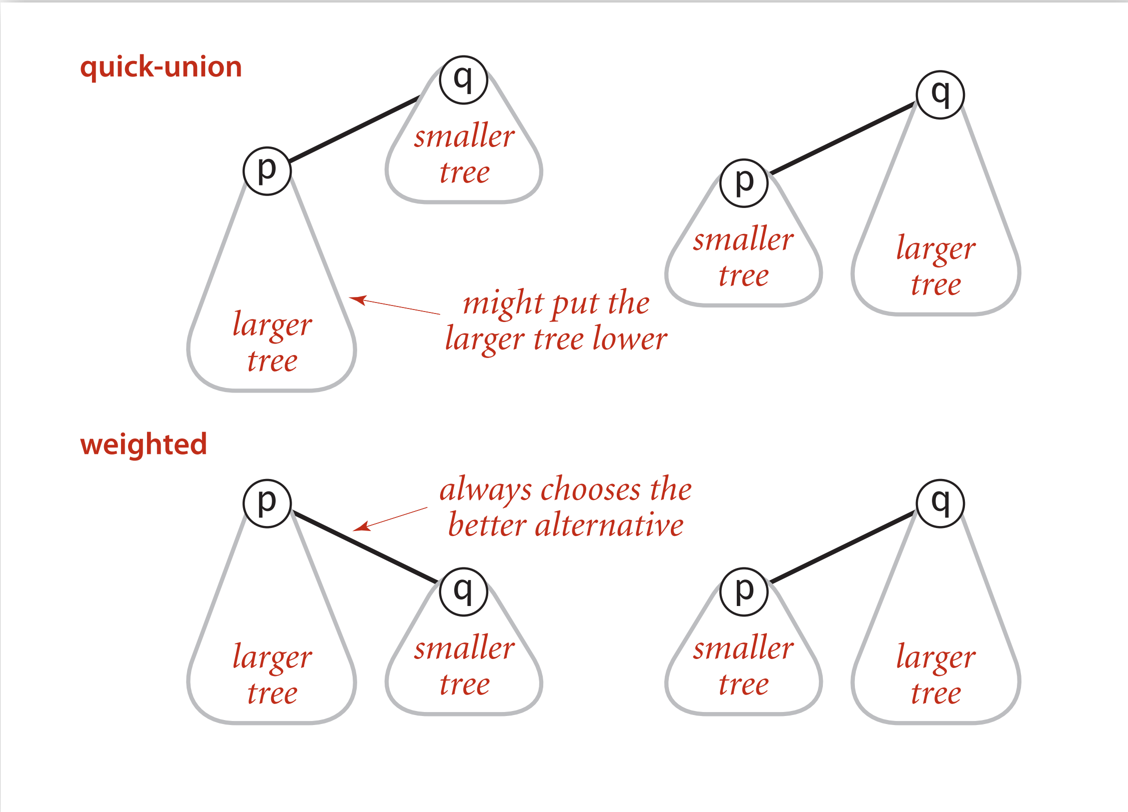
- Trees can get tall.
- Find too expensive (could be $N$ array accesses).

- dynamic connectivity
- quick find
- quick union
- **improvements**
- applications

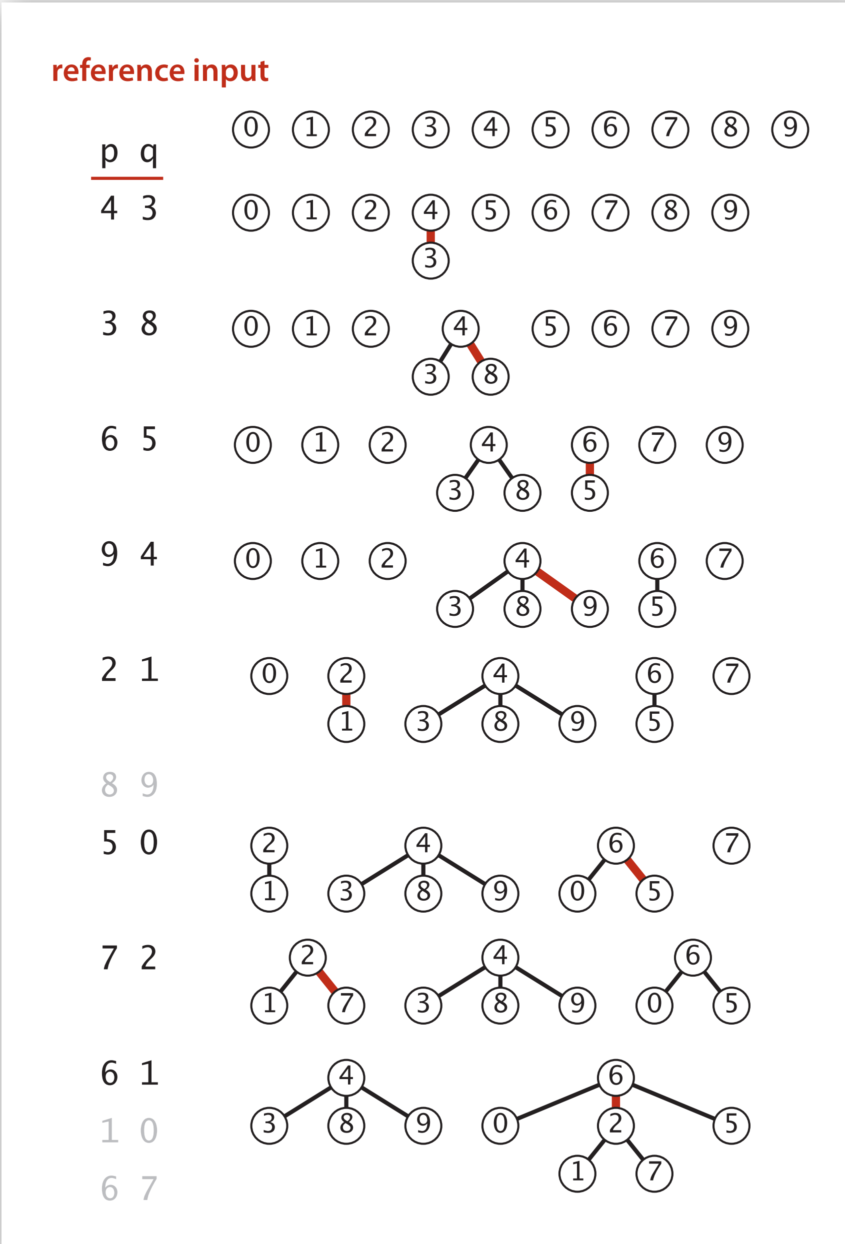
## Improvement 1: weighting

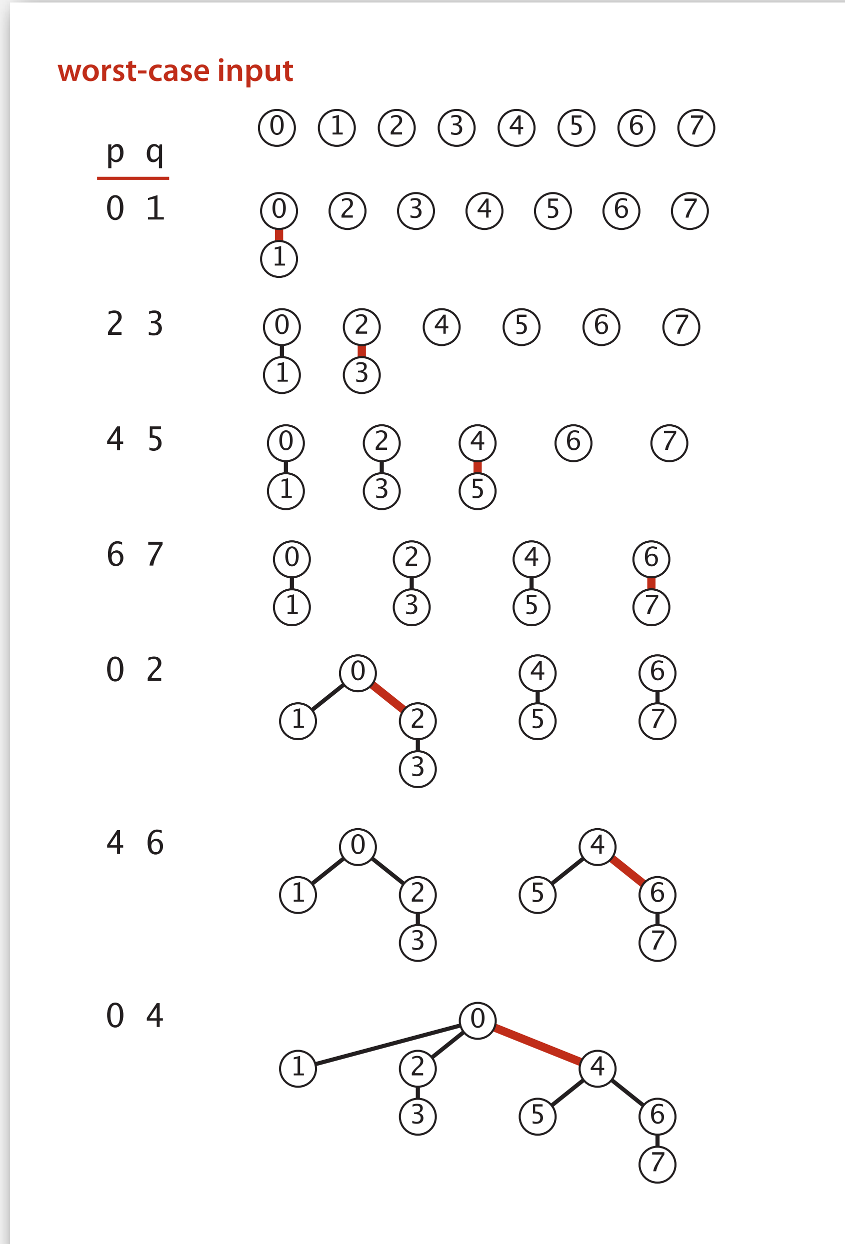
### Weighted quick-union.

- Modify quick-union to avoid tall trees.
- Keep track of size of each tree (number of objects).
- Balance by linking small tree below large one.

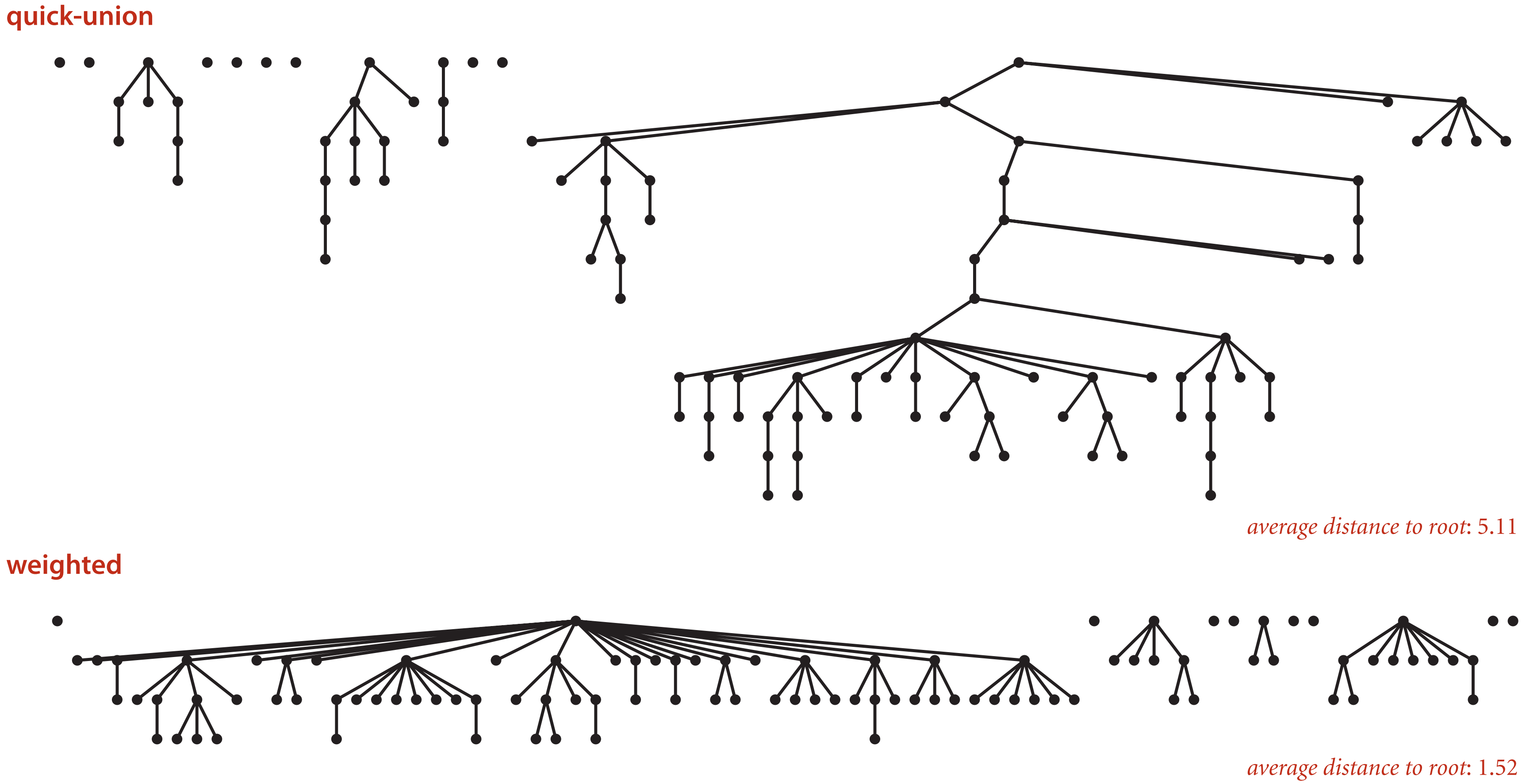


# Weighted quick-union examples



# Quick-union and weighted quick-union example




**Quick-union and weighted quick-union (100 sites, 88 `union()` operations)**

# Weighted quick-union: Java implementation

Data structure. Same as quick-union, but maintain extra array `sz[i]` to count number of objects in the tree rooted at `i`.

Find. Identical to quick-union.

```
return root(p) == root(q);
```

Union. Modify quick-union to:

- Merge smaller tree into larger tree.
- Update the `sz[]` array.

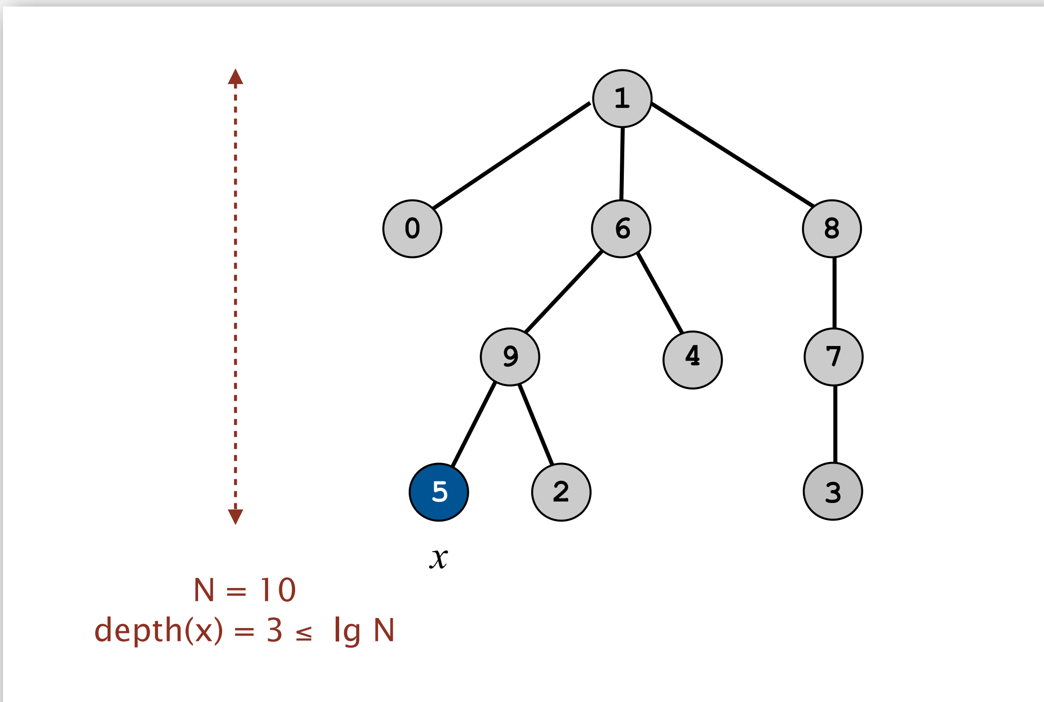
```
int i = root(p);
int j = root(q);
if  (sz[i] < sz[j]) { id[i] = j; sz[j] += sz[i]; }
else                { id[j] = i; sz[i] += sz[j]; }
```

# Weighted quick-union analysis

Running time.

- Find: takes time proportional to depth of $p$ and $q$.
- Union: takes constant time, given roots.

Proposition. Depth of any node $x$ is at most $\lg N$.



# Weighted quick-union analysis

**Running time.**

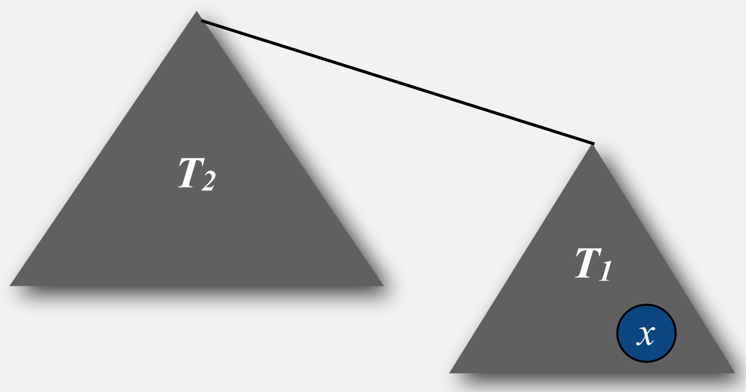
- Find: takes time proportional to depth of $p$ and $q$.
- Union: takes constant time, given roots.

**Proposition.** Depth of any node $x$ is at most $\lg N$.

**Pf.** When does depth of $x$ increase?

Increases by $1$ when tree $T_1$ containing $x$ is merged into another tree $T_2$.

- The size of the tree containing $x$ at least doubles since $|T_2| \geq |T_1|$.
- Size of tree containing $x$ can double at most $\lg N$ times. Why?



# Weighted quick-union analysis

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| algorithm | init | union | connected |
|---|---|---|---|
| quick-find | N | N | 1 |
| quick-union | N | N † | N |
| weighted QU | N | lg N † | lg N |

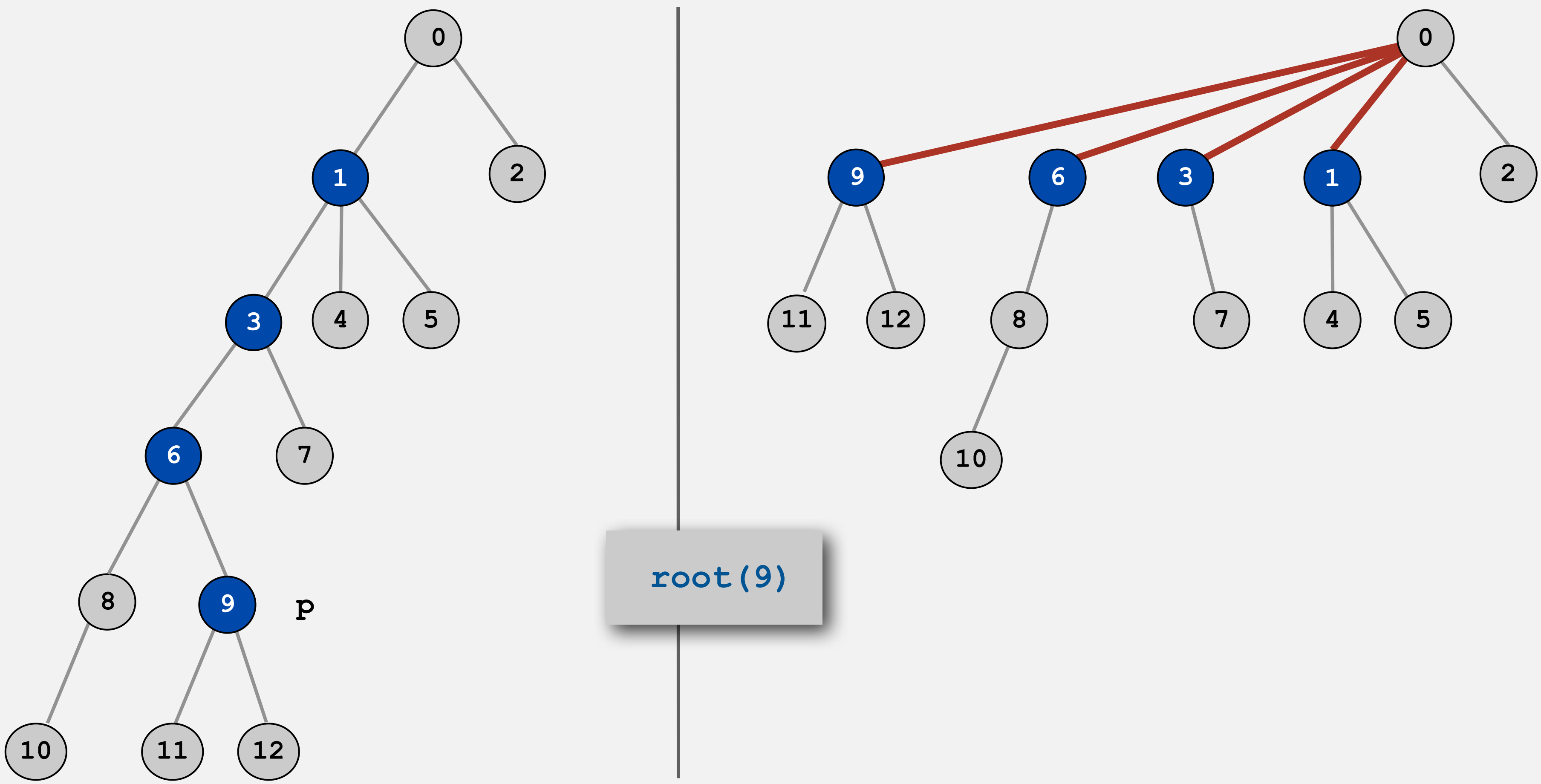
† includes cost of finding roots

Q. Stop at guaranteed acceptable performance?

A. No, easy to improve further.

# Improvement 2: path compression

Quick union with path compression. Just after computing the root of $p$, set the id of each examined node to point to that root.



# Path compression: Java implementation

Two-pass implementation: add second loop to `find()` to set the `id[]` of each examined node to the root.

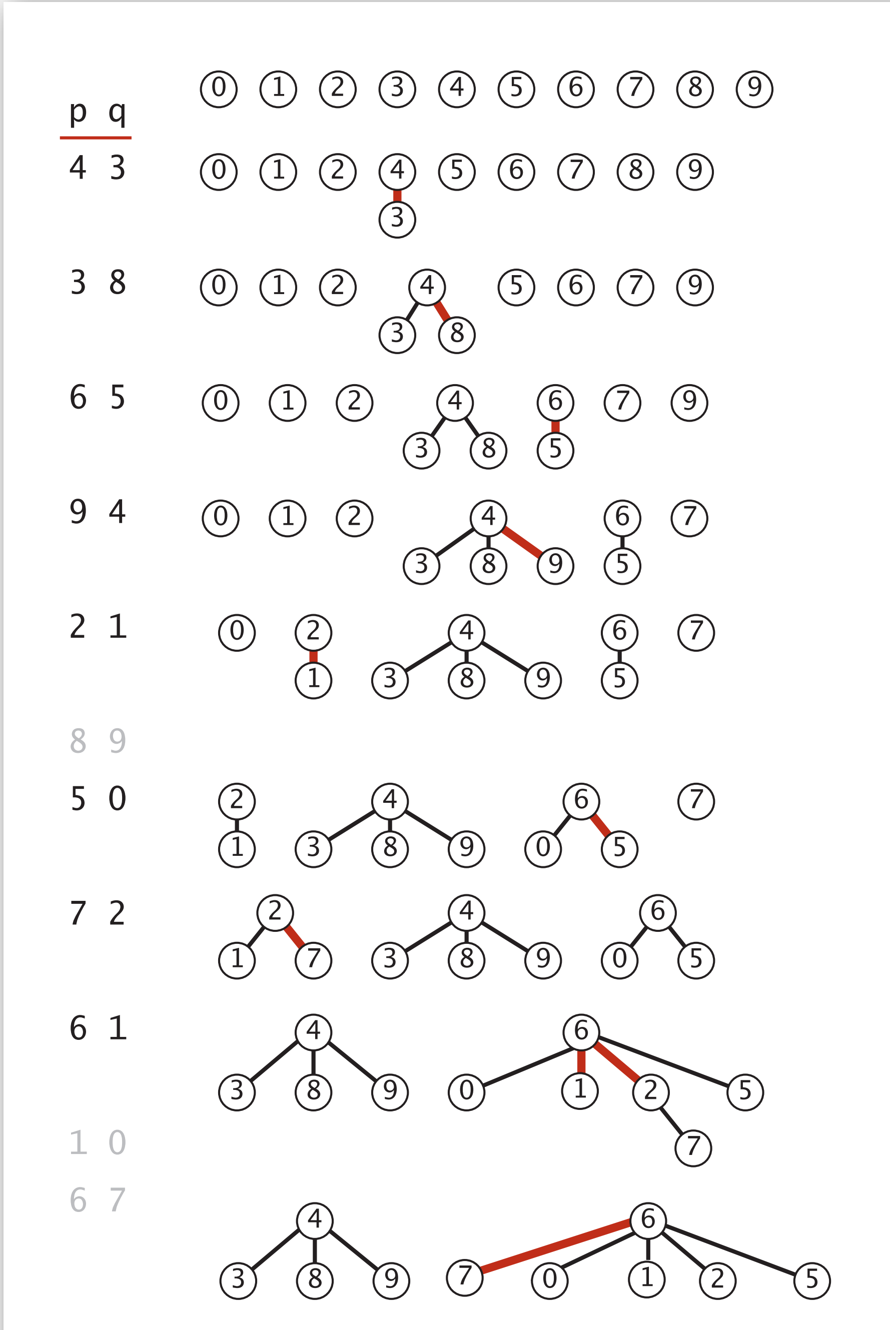
Simpler one-pass variant: Make every other node in path point to its grandparent (thereby halving path length).

```
private int root(int i)
{
   while (i != id[i])
   {
      id[i] = id[id[i]];
      i = id[i];
   }
   return i;
}
```

only one extra line of code !

In practice. No reason not to! Keeps tree almost completely flat.

# Weighted quick-union with path compression example




1 linked to 6 because of path compression

7 linked to 6 because of path compression

# Weighted quick-union with path compression: amortized analysis

**Proposition.** Starting from an empty data structure, any sequence of $M$ union–find operations on $N$ objects makes at most proportional to $N + M \lg^* N$ array accesses.

- Proof is very difficult.
- But the algorithm is still simple!
- Analysis can be improved to $N + M\,\alpha(M, N)$.

see COS 423

**Bob Tarjan**
**(Turing Award '86)**

## Linear-time algorithm for $M$ union-find ops on $N$ objects?

- Cost within constant factor of reading in the data.
- In theory, WQUPC is not quite linear.
- In practice, WQUPC is linear.

because $\lg^* N$ is a constant in this universe

| N | lg* N |
|---|---|
| 1 | 0 |
| 2 | 1 |
| 4 | 2 |
| 16 | 3 |
| 65536 | 4 |
| $2^{65536}$ | 5 |

lg* function

**Amazing fact.** No linear-time algorithm exists.

in "cell-probe" model of computation

# Summary

Bottom line. WQUPC makes it possible to solve problems that could not otherwise be addressed.

| algorithm | worst-case time |
|---|---|
| quick-find | M N |
| quick-union | M N |
| weighted QU | N + M log N |
| QU + path compression | N + M log N |
| weighted QU + path compression | N + M lg* N |

**M union-find operations on a set of N objects**

Ex. [$10^9$ unions and finds with $10^9$ objects]

- WQUPC reduces time from 30 years to 6 seconds.
- Supercomputer won't help much; good algorithm enables solution.

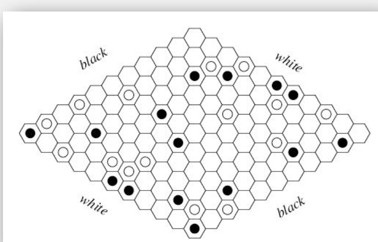
- dynamic connectivity
- quick find
- quick union
- improvements
- **applications**

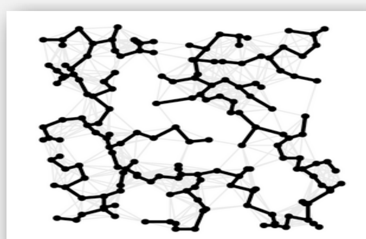
# Union-find applications

- Percolation.
- Games (Go, Hex).

✓ Dynamic connectivity.

- Least common ancestor.
- Equivalence of finite state automata.
- Hoshen-Kopelman algorithm in physics.
- Hinley-Milner polymorphic type inference.
- Kruskal's minimum spanning tree algorithm.
- Compiling equivalence statements in Fortran.
- Morphological attribute openings and closings.
- Matlab's `bwlabel()` function in image processing.

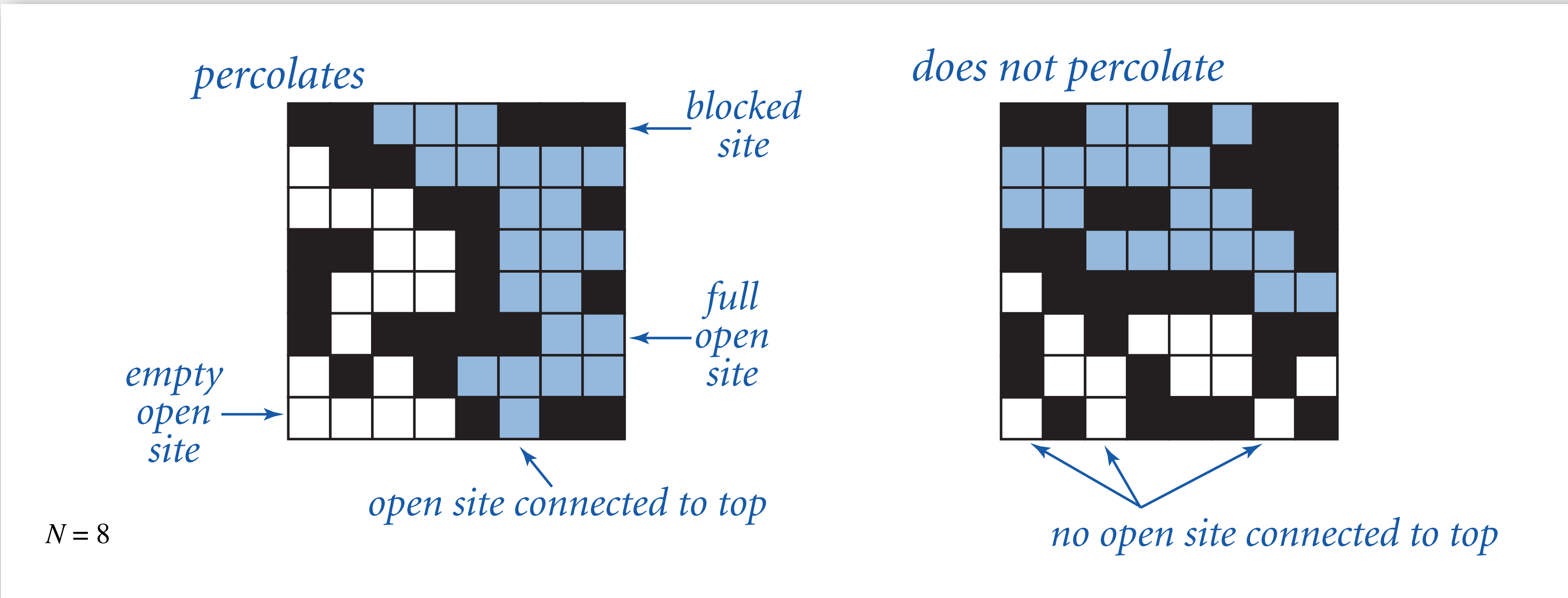


# Percolation

A model for many physical systems:

- $N$-by-$N$ grid of sites.
- Each site is open with probability $p$ (or blocked with probability $1 - p$).
- System **percolates** iff top and bottom are connected by open sites.



# Percolation

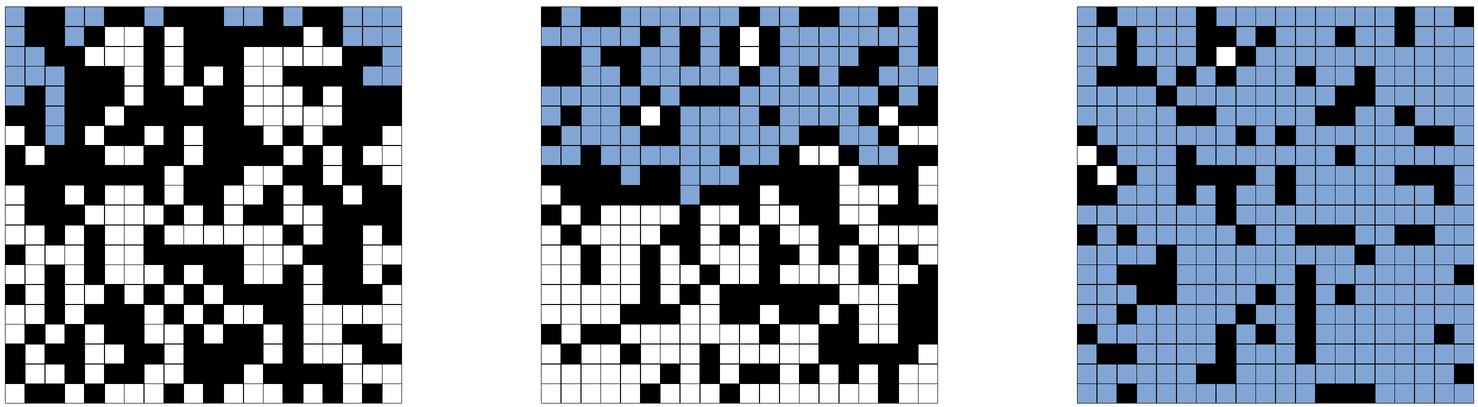
A model for many physical systems:

- $N$-by-$N$ grid of sites.
- Each site is open with probability $p$ (or blocked with probability $1 - p$).
- System percolates iff top and bottom are connected by open sites.

| model | system | vacant site | occupied site | percolates |
|---|---|---|---|---|
| electricity | material | conductor | insulated | conducts |
| fluid flow | material | empty | blocked | porous |
| social interaction | population | person | empty | communicates |

# Likelihood of percolation

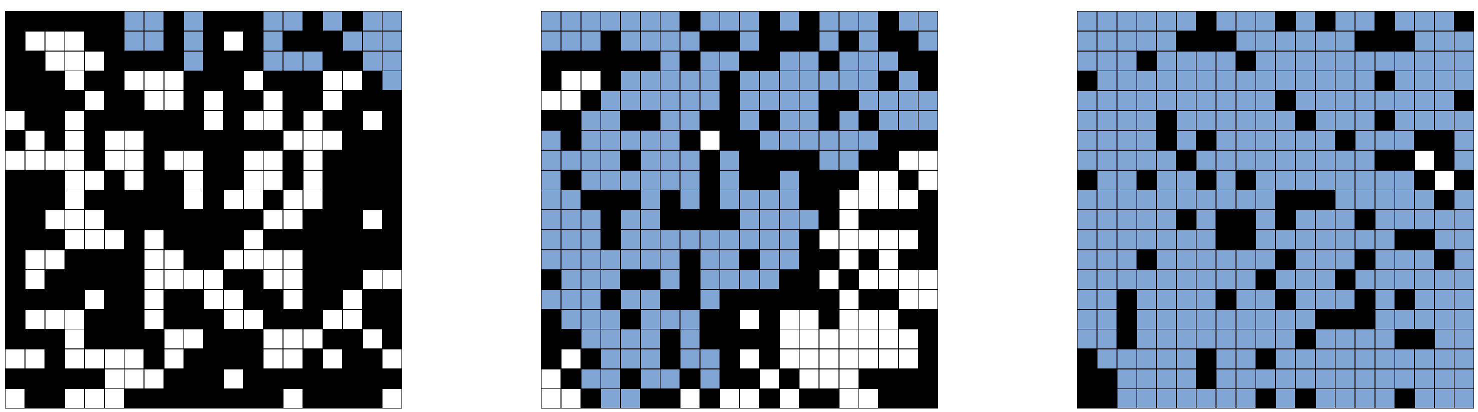
Depends on site vacancy probability $p$.



p low (0.4)
does not percolate

p medium (0.6)
percolates?

p high (0.8)
percolates

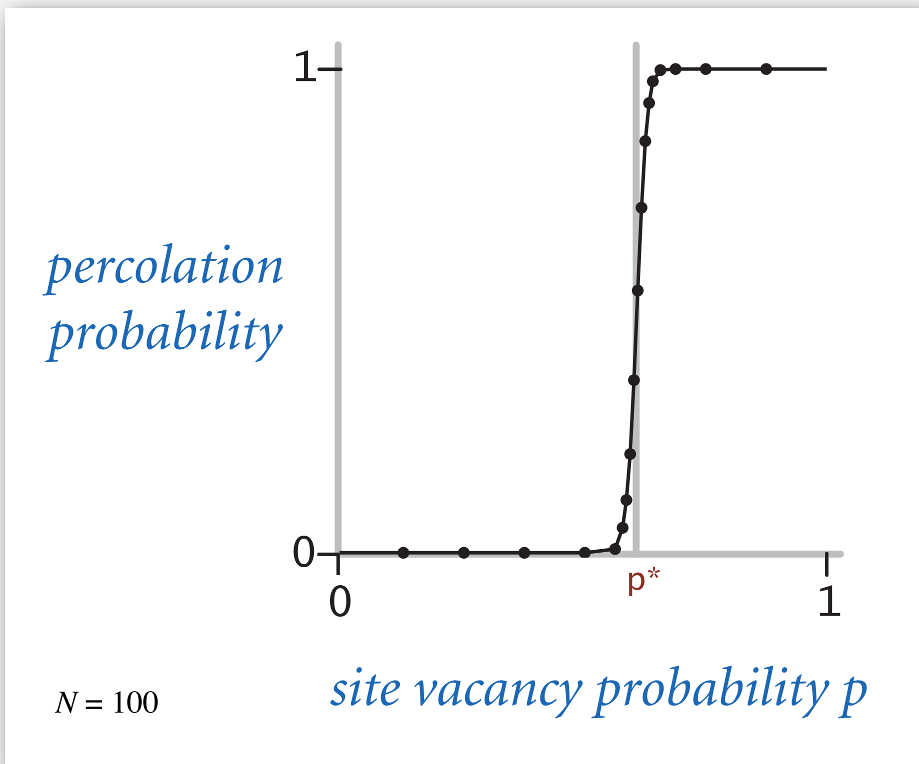


# Percolation phase transition

When $N$ is large, theory guarantees a sharp threshold $p^*$.

- $p > p^*$: almost certainly percolates.
- $p < p^*$: almost certainly does not percolate.

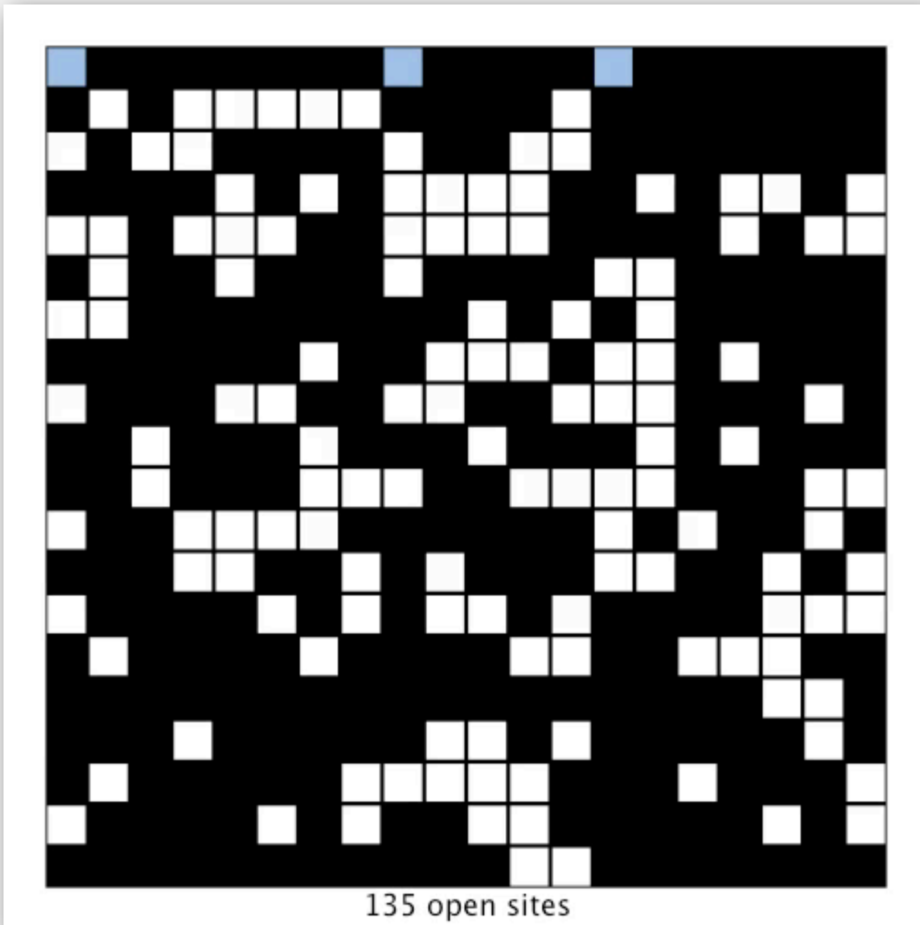
Q. What is the value of $p^*$ ?

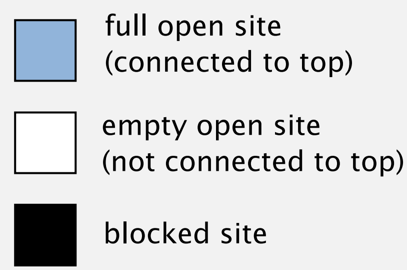


# Monte Carlo simulation

- Initialize $N$-by-$N$ whole grid to be blocked.
- Declare random sites open until top connected to bottom.
- Vacancy percentage estimates $p^*$.

$N = 20$

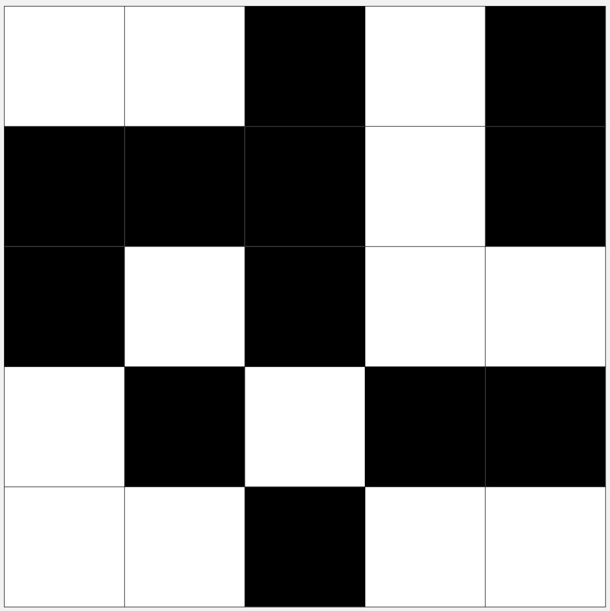


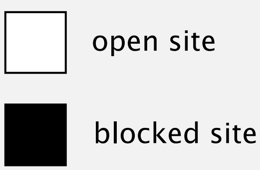



# Dynamic connectivity solution to estimate percolation threshold

**Q.** How to check whether an $N$-by-$N$ system percolates?

$N = 5$

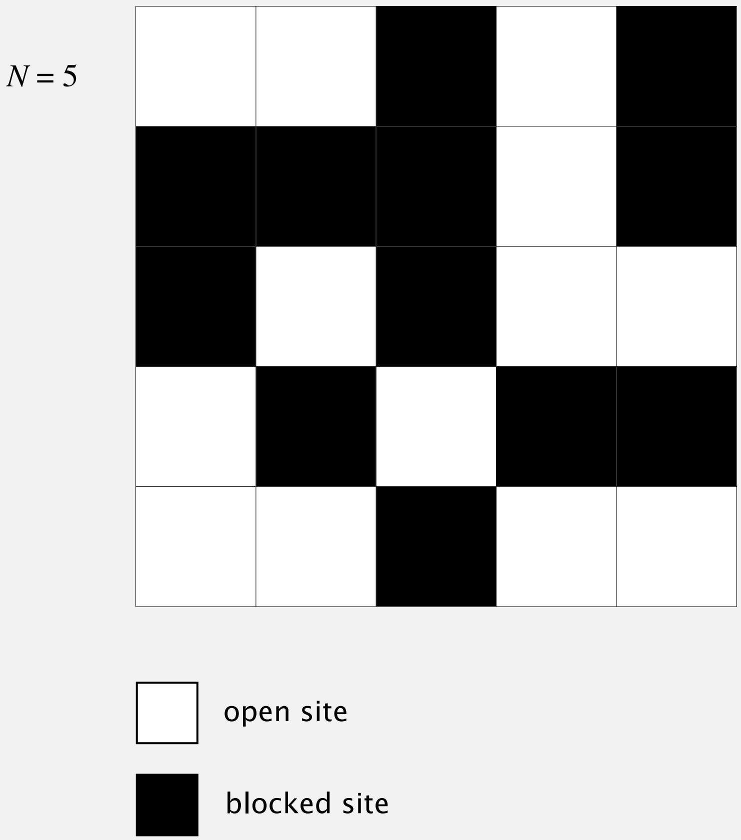


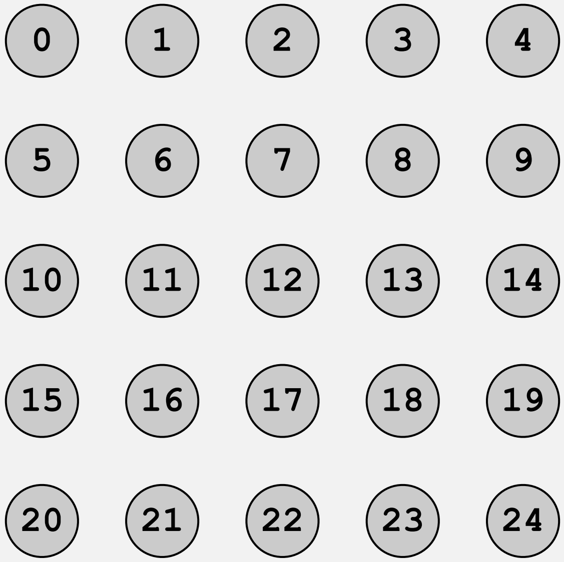


# Dynamic connectivity solution to estimate percolation threshold

**Q.** How to check whether an $N$-by-$N$ system percolates?

- Create an object for each site and name them $0$ to $N^2 - 1$.

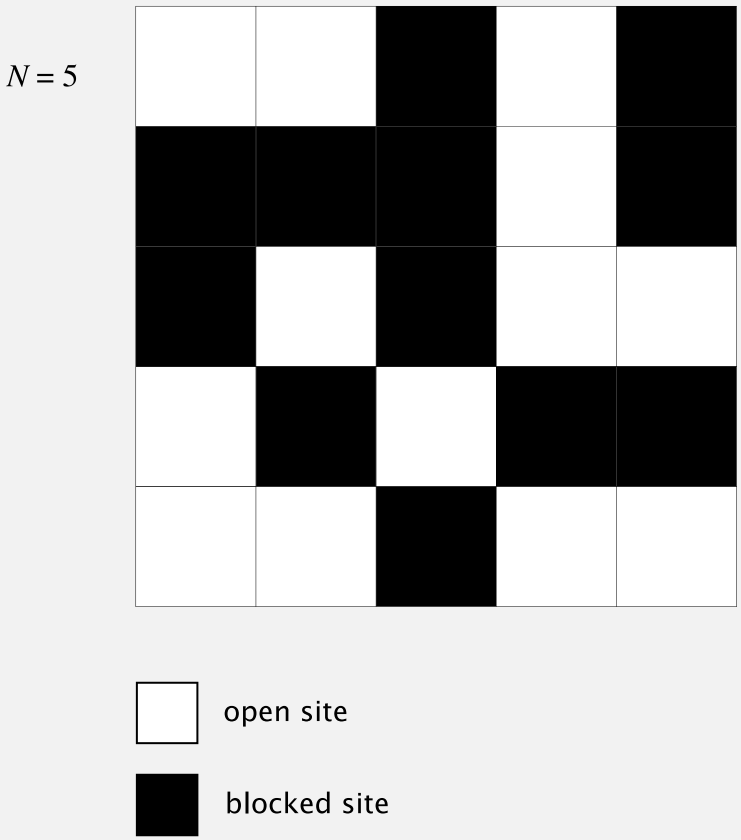


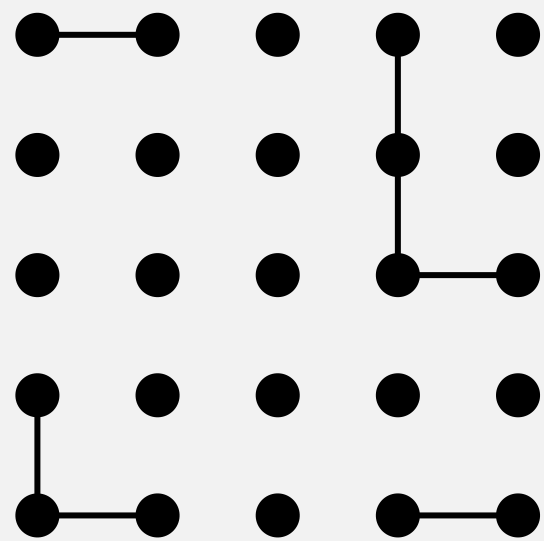



# Dynamic connectivity solution to estimate percolation threshold

**Q.** How to check whether an $N$-by-$N$ system percolates?

- Create an object for each site and name them $0$ to $N^2 - 1$.
- Sites are in same component if connected by open sites.



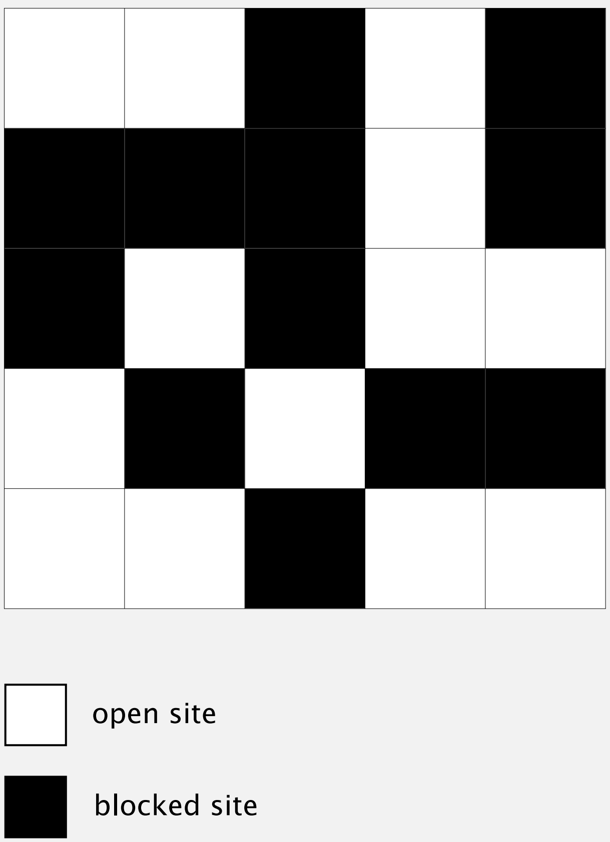
# Dynamic connectivity solution to estimate percolation threshold

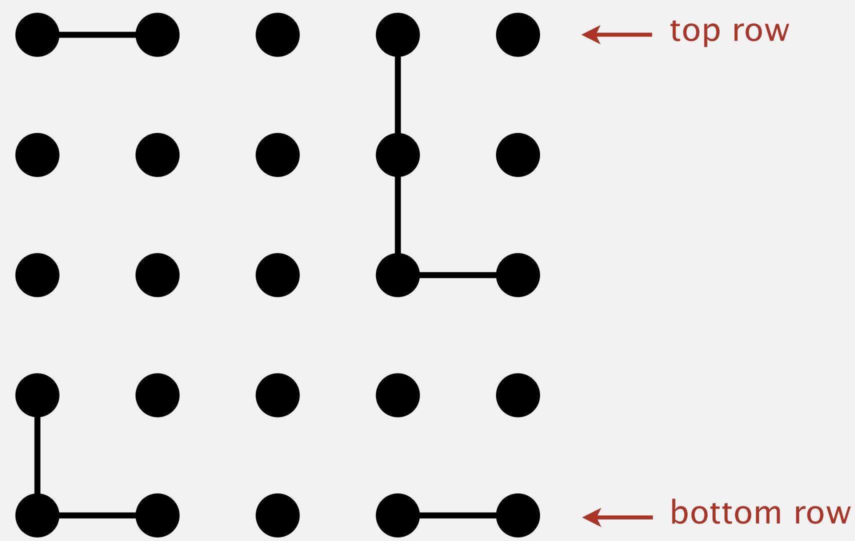
**Q.** How to check whether an $N$-by-$N$ system percolates?

- Create an object for each site and name them $0$ to $N^2 - 1$.
- Sites are in same component if connected by open sites.
- Percolates iff any site on bottom row is connected to site on top row.

brute-force algorithm: $N^2$ calls to connected()

$N = 5$



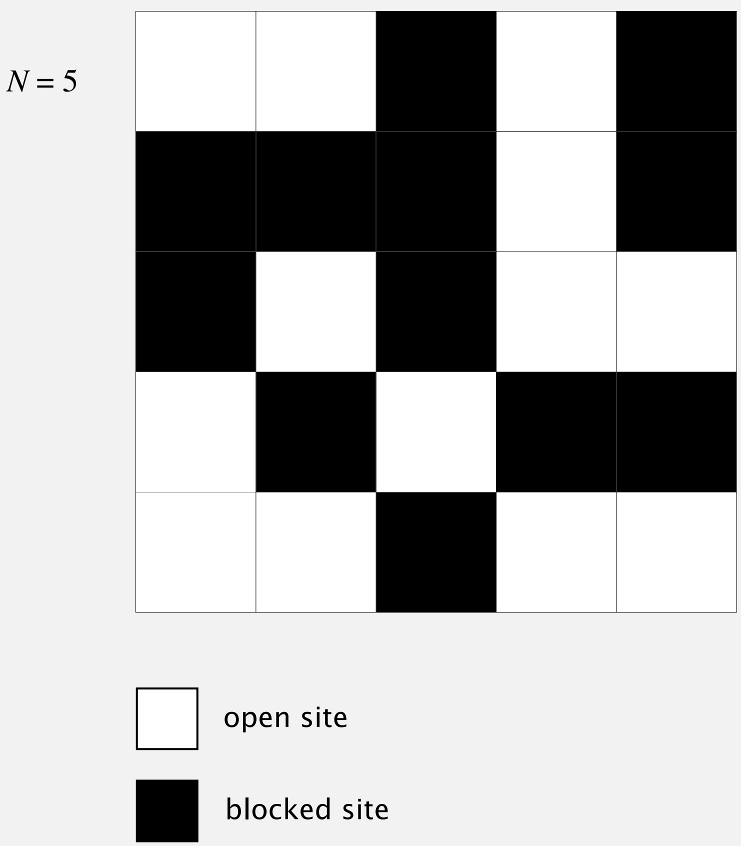



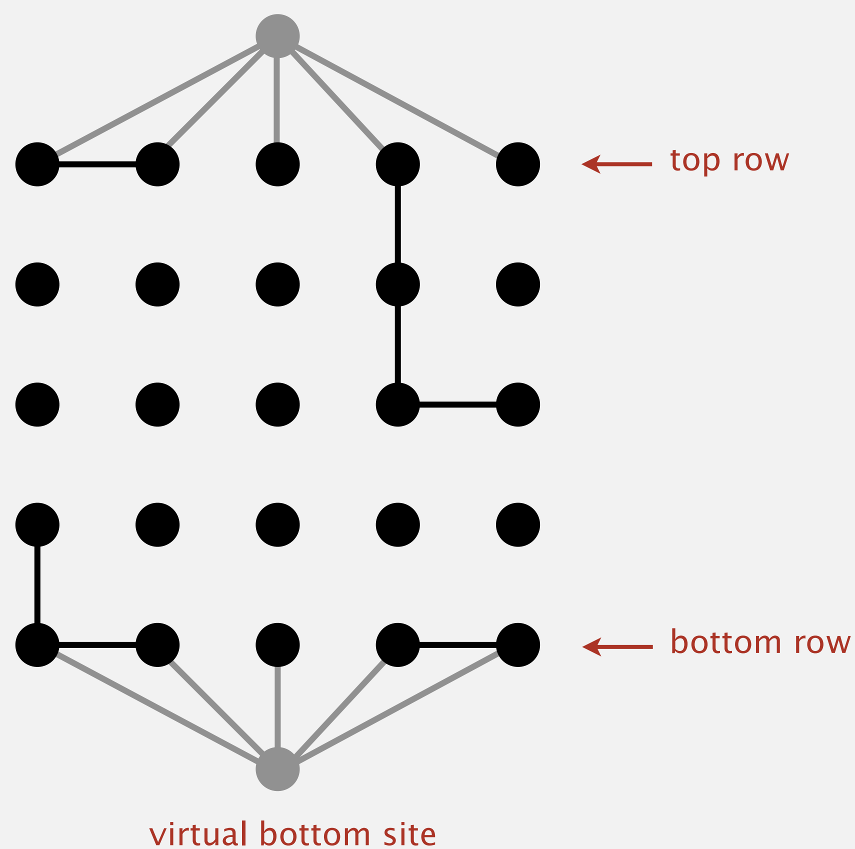
# Dynamic connectivity solution to estimate percolation threshold

**Clever trick.** Introduce two virtual sites (and connections to top and bottom).

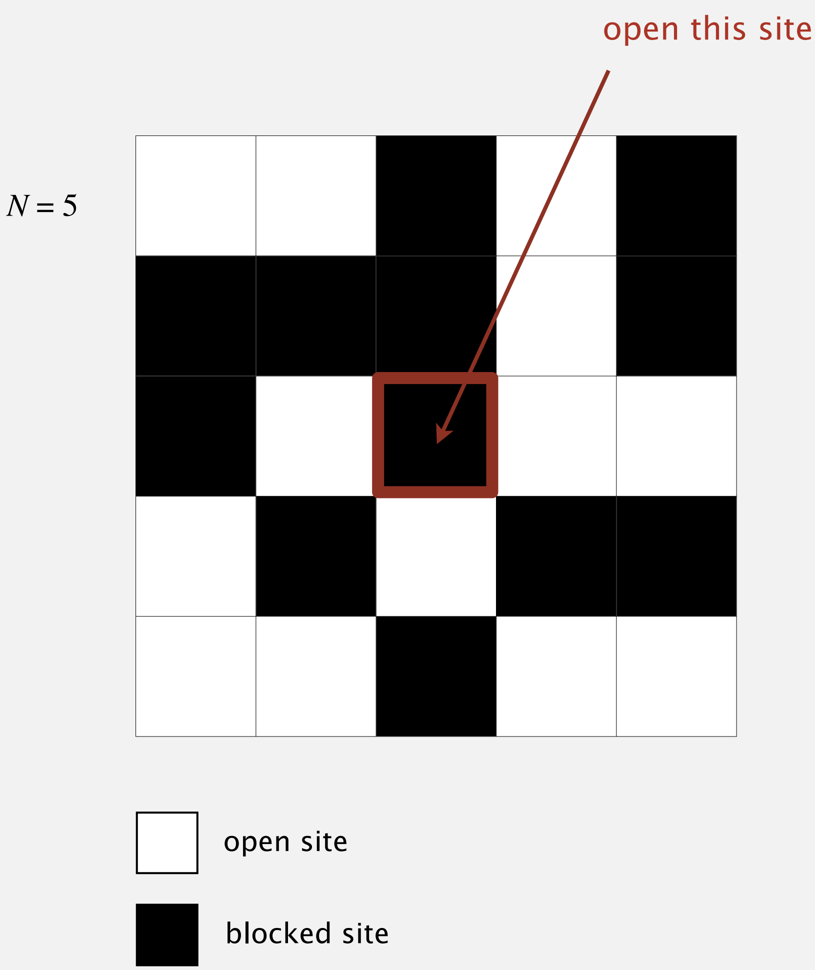
- Percolates iff virtual top site is connected to virtual bottom site.

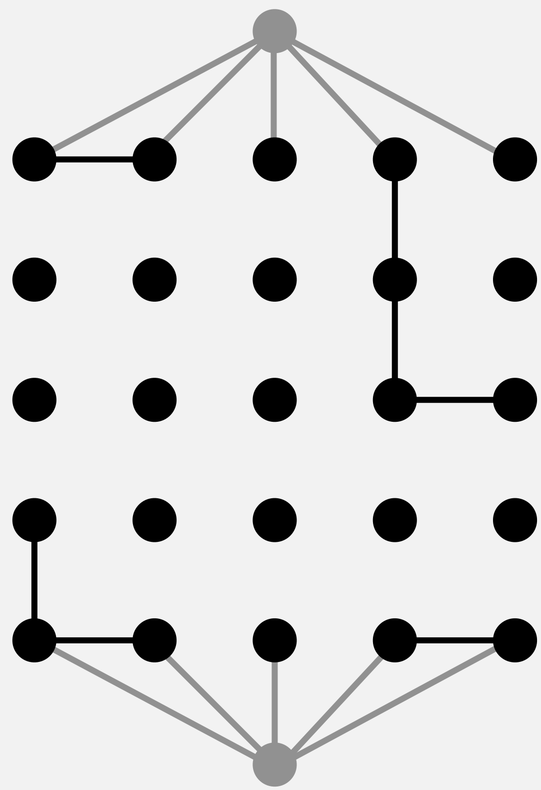
efficient algorithm: only 1 call to connected()



**Q.** How to model as dynamic connectivity problem when opening a new site?



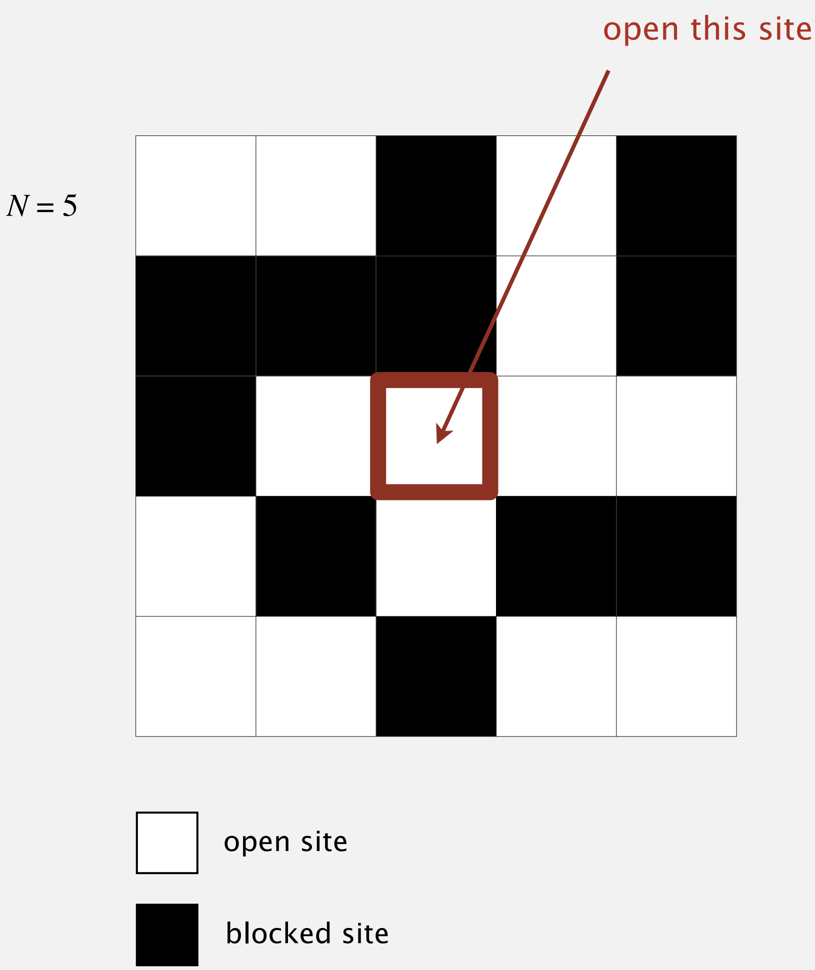



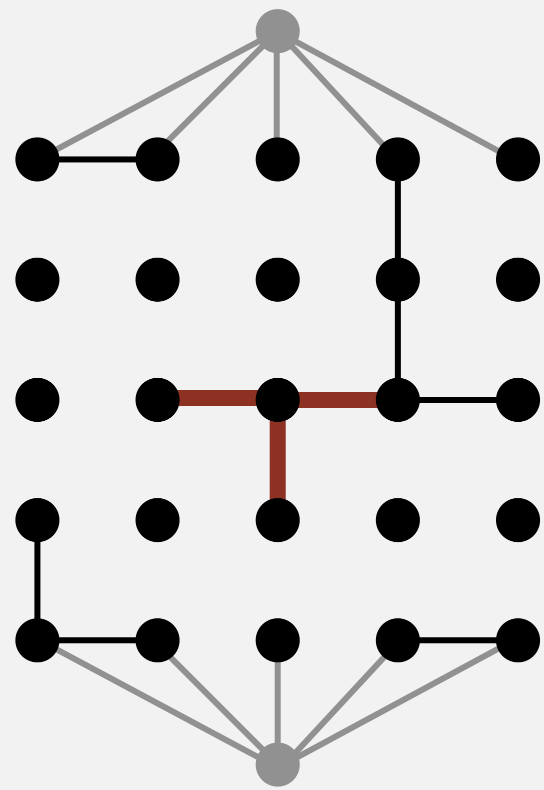
# Dynamic connectivity solution to estimate percolation threshold

**Q.** How to model as dynamic connectivity problem when opening a new site?

**A.** Connect newly opened site to all of its adjacent open sites.

up to 4 calls to union()



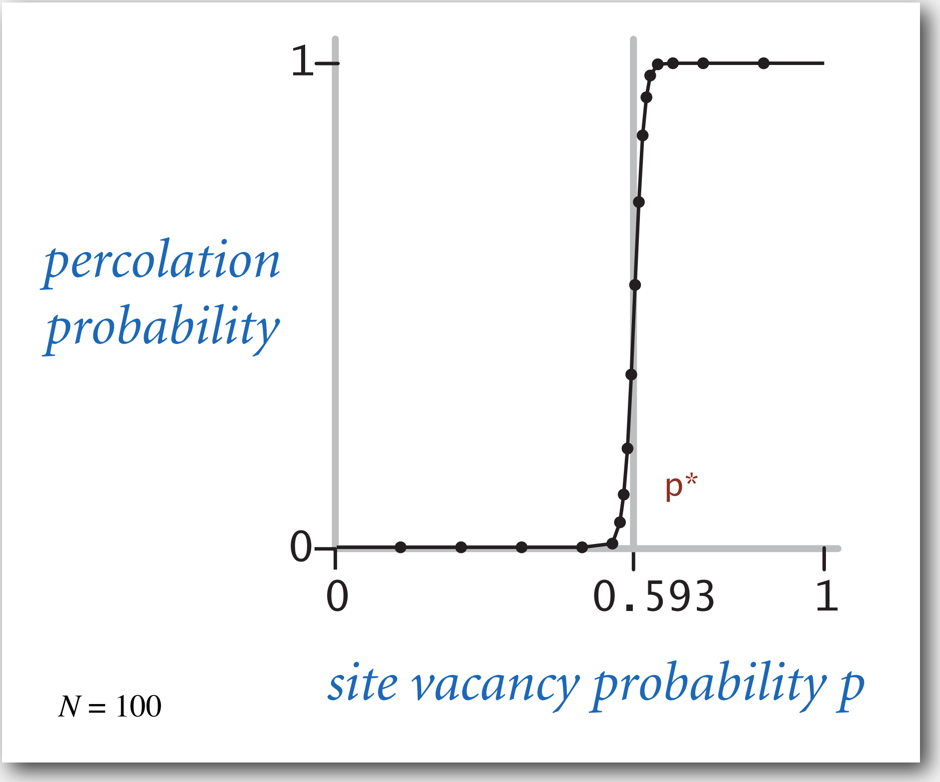



## Percolation threshold

Q. What is percolation threshold $p^*$ ?

A. About 0.592746 for large square lattices.

constant know only via simulation




Fast algorithm enables accurate answer to scientific question.

# Subtext of today's lecture (and this course)

Steps to developing a usable algorithm.

- Model the problem.
- Find an algorithm to solve it.
- Fast enough? Fits in memory?
- If not, figure out why.
- Find a way to address the problem.
- Iterate until satisfied.

The scientific method.

Mathematical analysis.