p-ADIC COUPLING OF MOCK MODULAR FORMS AND SHADOWS

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ABSTRACT. A *mock modular form* is the holomorphic part of a harmonic Maass form f. The non-holomorphic part of f is a period integral of its *shadow*, a cusp form g. A direct method for relating the coefficients of shadows and mock modular forms is not known. The fact that a shadow can be cast by infinitely many mock modular forms, and the expected transcendence of generic mock modular forms pose serious obstructions to this problem. We solve these problems when the shadow is an integer weight newform. Our solution is p-adic, and it relies on our definition of an algebraic *regularized mock modular form.* As an application, we consider the modular solution to the cubic *arithmetic-geometric mean.*

1. Introduction

The theory of harmonic weak Maass forms[1] [1, 2, 3], which explains Ramanujan's mock theta functions [3, 4, 5, 6], relies on a correspondence between harmonic Maass forms and cusp forms. For example, Ramanujan's mock theta function (note $q := e^{2\pi iz}$)

$$f(q) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2} = 1 + q - \cdots - 53q^{24} + \cdots + 19618q^{101} - \cdots$$

is coupled to the weight 3/2 cuspidal theta function

$$\Theta(z) := \sum_{n=-\infty}^{\infty} (6n+1)q^{\frac{3}{2}\left(n+\frac{1}{6}\right)^2} = q^{\frac{1}{24}} - 5q^{\frac{25}{24}} + 7q^{\frac{49}{24}} - 11q^{\frac{121}{24}} + 13q^{\frac{169}{24}} - \ldots$$

by Zwegers's [5] harmonic Maass form

$$M_f(z) := q^{-1}f(q^{24}) + \frac{i\sqrt{3}}{3}\int_{-24\bar{z}}^{i\infty} \frac{\Theta(\tau)}{\sqrt{-i(\tau+24z)}}\, d\tau.$$

Following Zagier [4], we refer to $q^{-1}f(q^{24})$ as a *mock modular form*, and $\Theta(z)$ as its *shadow*.

We do not know a simple relationship between the coefficients of $f(q)$ and $\Theta(z)$. More generally, we have the following natural problem.

Problem. *Relate the coefficients of a mock modular form to the coefficients of its shadow.*

We solve this problem when the shadow is an integer weight newform.

For an integer $k \geq 2$, let $H_{2-k}(\Gamma_0(N), \chi)$ be the space of weight $2-k$ harmonic Maass forms on $\Gamma_0(N)$ with Nebentypus χ. Every $f(z) \in H_{2-k}(\Gamma_0(N), \chi)$ may be expressed as

$$f(z) = f^+(z) + f^-(z),$$

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[1]Throughout we will use the terminology *harmonic Maass form.*

where f^+ is the *holomorphic part*, the so-called *mock modular form*, and where f^- is its *non-holomorphic part*. These non-holomorphic parts are power series in q and the incomplete Gamma function evaluated at arguments in $\mathrm{Im}(z)$ and k (for example, see Section 3 of [2]).

The role of these non-holomorphic parts in the theory of harmonic Maass forms is revealed by the differential operator $\xi = \xi_{2-k} := 2iy^{2-k}\overline{\frac{\partial}{\partial \bar{z}}}$. It defines a surjective map

$$\xi\colon H_{2-k}(\Gamma_0(N),\chi) \longrightarrow S_k(\Gamma_0(N),\overline{\chi}),$$

where $S_k(\Gamma_0(N),\overline{\chi})$ denotes the space of weight k cusp forms on $\Gamma_0(N)$ with Nebentypus $\overline{\chi}$. We refer to $\xi(f) = \xi(f^-) = g \in S_k(\Gamma_0(N),\overline{\chi})$ as the *shadow* of f^+.

Remark 1. Since $\ker(\xi) = M^!_{2-k}(\Gamma_0(N),\chi) \subset H_{2-k}(\Gamma_0(N),\chi)$, the subspace of weakly holomorphic modular forms, different mock modular forms can cast the same shadow.

For the remainder of the paper, we fix a normalized (i.e. $b_g(1) = 1$) newform

$$g(z) = \sum_{n=1}^{\infty} b_g(n)q^n \in S_k(\Gamma_0(N),\chi) \tag{1.1}$$

with weight $2 \leq k \in \mathbb{Z}$ and real Nebentypus[2] χ. Let $E_g(z)$ be its *Eichler integral*

$$E_g(z) := \sum_{n=1}^{\infty} b_g(n)n^{1-k}q^n, \tag{1.2}$$

and let K_g be the number field obtained by adjoining to $\mathbb{Q}$ the coefficients $b_g(n)$.

Remark 2. For convenience we have assumed that χ is real so that ξ preserves Nebentypus.

Following [7], we say that $f(z) \in H_{2-k}(\Gamma_0(N),\chi)$ is *good* for the newform

$$g^c(z) := \overline{g(-\bar{z})} = \sum_{n=1}^{\infty} \overline{b_g(n)}q^n \in S_k(\Gamma_0(N),\chi)$$

if it satisfies the following:

(1) The principal part of f at the cusp ∞ belongs to $K_g[q^{-1}]$.
(2) The principal parts of f at other cusps (if any) are constant.
(3) We have $\xi(f) = \frac{g^c}{\langle g^c, g^c\rangle}$, where $\langle g^c, g^c\rangle$ is the Petersson product of $g^c(z)$ with itself.

Remark 3. The existence of f which are good for g^c is guaranteed by Proposition 5.1 of [7].

We now fix an $f \in H_{2-k}(\Gamma_0(N),\chi)$ which is good for g^c. Its mock modular form is

$$f^+ = \sum_{n \gg -\infty} c_f(n)q^n. \tag{1.3}$$

Bruinier, Rhoades, and the third author [7] proved that f^+ has algebraic coefficients if g has complex multiplication (CM). Otherwise, we expect a completely different phenomenon. For example, for $g = \Delta$, the unique normalized weight 12 cusp form on $\mathrm{SL}_2(\mathbb{Z})$, we have [8]

$$11!f^+ \sim 11!q^{-1} - \frac{2615348736000}{691} - 73562460235.68364q - 929026615019.11308q^2 - \dots. \tag{1.4}$$

After the first two coefficients, the coefficients appear (see [8]) to be transcendental.

[2]We suppress χ from the notation when it is trivial.

Conjecture. *Assume the notation and hypotheses above. The mock modular form f^+ has some transcendental coefficients if and only if its shadow g does not have CM.*

Despite the ambiguity concerning the algebraicity of mock modular forms, we show that f^+ may be regularized in a simple way to obtain an algebraic series.

Theorem 1.1. *Assume the notation and hypotheses above. If α is a complex number for which $\alpha - c_f(1) \in K_g$, then the coefficients of*

$$\mathcal{F}_\alpha := f^+(z) - \alpha E_g(z) = \sum_{n \gg -\infty} c_f(n) q^n - \alpha \sum_{n=1}^{\infty} b_g(n) n^{1-k} q^n$$

are in K_g. In particular, the transcendence degree of $K_g(c_f(n))$ over K_g is at most one.

Remark 4. Obviously, one may always let $\alpha := c_f(1)$ in Theorem 1.1.

Example. For $g = \Delta$, if we let $\alpha := c_f(1)$, then Theorem 1.1 implies that $\mathcal{F}_\alpha$ has $\mathbb{Q}$-rational coefficients. Numerically, we indeed find that

$$11!\mathcal{F}_\alpha = 11!q^{-1} - \frac{2615348736000}{691} - 929888675100q^2 - \frac{80840909811200}{9}q^3 - \dots . \tag{1.5}$$

Now we fix a complex number α for which $\alpha - c_f(1) \in K_g$. If $\mathcal{F}_\alpha$ is as in Theorem 1.1, then we refer to this K_g-rational power series as a *regularized mock modular form*. We shall employ these regularizations to couple mock modular forms to their shadows.

To this end, let p be prime. We fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, along with an embedding of $\overline{\mathbb{Q}}$ into $\overline{\mathbb{Q}}_p$. Furthermore, we embed $\overline{\mathbb{Q}}_p$ into its completion $\mathbb{C}_p$. These embeddings determine an extension of the p-adic valuation to K_g. We denote by $\mathrm{ord}_p : \mathbb{C}_p \to \mathbb{Q}$ the p-adic order normalized so that $\mathrm{ord}_p(p) = 1$.

Remark 5. It will be clear from our results that the Fourier coefficients of $\mathcal{F}_\alpha$ may have unbounded negative p-adic order (also see Remark 5 of [7]).

To relate $\mathcal{F}_\alpha$ to g, we use the operator $D := \frac{1}{2\pi i} \cdot \frac{d}{dz}$, and we let

$$F_\alpha := D^{k-1}\mathcal{F}_\alpha = \sum_{n \gg -\infty} c_\alpha(n) q^n. \tag{1.6}$$

By Theorem 1.1 of [7], combined with the obvious fact that $D^{k-1}E_g = g$, it follows that

$$F_\alpha = \sum_{n \gg -\infty} \left(n^{k-1} c_f(n) - \alpha b_g(n)\right) q^n$$

is a weight k weakly holomorphic modular form in $M_k^!(\Gamma_0(N), \chi)$. We shall iteratively apply Atkin's $U := U(p)$ operator to F_α to couple mock modular forms with their shadows. This operator is defined on formal power series by

$$\left(\sum a(n) q^n\right) \mid U = \sum a(pn) q^n. \tag{1.7}$$

To state our result, let β, β' be the roots of the equation

$$X^2 - b_g(p)X + \chi(p)p^{k-1} = (X - \beta)(X - \beta') \tag{1.8}$$

ordered so that $\mathrm{ord}_p(\beta) \leq \mathrm{ord}_p(\beta')$. We then define the cusp form $\breve{g}$ by

$$\breve{g} = \sum_{n=1}^{\infty} \breve{b}_g(n) q^n := g(z) - \beta^{-1}\chi(p)p^{k-1}g(pz). \tag{1.9}$$

If $p \nmid N$, then $\breve{g} \in S_k(\Gamma_0(pN), \chi)$, and if $p \mid N$ then $\breve{g} = g$ since $\chi(p) = 0$.

We now solve the motivating problem for g^c by relating the coefficients of g and $\breve{g}$ to the coefficients of the mock modular form f and the regularized mock modular form $\mathcal{F}_\alpha$. If g has CM, we define

$$G_\alpha(z) := D^{k-1}\left(f^+(z) - \alpha E_g(pz)\right) = \sum_{n \gg -\infty} d_\alpha(n) q^n$$

for $\alpha \in K_g$.

Theorem 1.2. *Assume the notation and hypotheses above.*

(1) *Suppose that $p \nmid N$ and $\mathrm{ord}_p(\beta) \neq (k-1)/2$, or $p \mid N$ and $\beta \neq 0$. For all but at most one choice of α with $\alpha - c_f(1) \in K_g$, we have that*

$$\breve{g} = \lim_{w \to +\infty} \frac{F_\alpha \mid U(p^w)}{c_\alpha(p^w)}.$$

(2) *Suppose that g has CM. If p is inert in the field of complex multiplication, then for all but at most one choice of $\alpha \in K_g$ we have that*

$$g = \lim_{w \to +\infty} \frac{G_\alpha \mid U(p^{2w+1})}{d_\alpha(p^{2w+1})}.$$

Remark 6. We comment on the limits in Theorem 1.2. It can happen that some of the coefficients appearing in the denominators of these formulas vanish. The proof of Theorem 1.2 will show that there are at most finitely many w for which these denominators vanish.

Remark 7. In the case of trivial tame level (i.e. N is a power of p) and trivial Nebentypus χ, the space of p-ordinary cusp forms is empty if $p \leq 7$, or $k = 4, 6, 8, 10, 14 \mod (p-1)$. Thus g is necessarily non-p-ordinary (i.e. $\mathrm{ord}_p(\beta) > 0$), and the proof of Theorem 1.2 and Proposition 2.2 below implies Theorem 1.5 of [7] that certain series are p-adic modular forms.

Remark 8. The proof of Theorem 1.2 can break down for one exceptional α. For example, if g has CM, then $\alpha = 0$ can be exceptional when p is a prime which splits in the field of complex multiplication. These exceptional cases are of interest, and they correspond to situations where one directly obtains p-adic modular forms without iteration.

Example. Theorem 1.2 implies infinitely many systematic congruences. For $g = \Delta$ and $p = 3$, we have that $\mathrm{ord}_3(\beta) = 2$, and also that $\Delta \equiv \breve{\Delta} \pmod{3^9}$. Using (1.4) and (1.5), we find that the $w = 1$ term in Theorem 1.2 (1) numerically gives

$$-\frac{D^{11}\left(f^+ - c_f(1)\sum_{n=1}^{\infty} \tau(n) n^{-11} q^n\right) \mid U(3)}{39862705122} \equiv \Delta \pmod{27}.$$

We conclude with an application of these results. We consider the modular forms which arise in the *cubic arithmetic-geometric mean* (AGM) of Borwein and Borwein [9]. The cubic AGM is the concordant limit of two sequences defined by

$$M_1(A, B) := \frac{A + 2B}{3} \quad \text{and} \quad M_2(A, B) := \sqrt[3]{\frac{B(A^2 + AB + B^2)}{3}}.$$

Suppose that $b_1 := b \leq a =: a_1$ are two positive real numbers. For positive n let

$$a_{n+1} := M_1(a_n, b_n) \quad \text{and} \quad b_{n+1} := M_2(a_n, b_n).$$

The cubic AGM of a and b is the coincidental limit

$$\mathrm{AGM}_3(a, b) := \lim_{n\to+\infty} a_n = \lim_{n\to+\infty} b_n.$$

For $0 < x < 1$, it turns out that $\mathrm{AGM}_3(1, x)$ is given by the hypergeometric formula

$$\frac{1}{\mathrm{AGM}_3(1,x)} = {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1-x^3\right), \tag{1.10}$$

where ${}_2F_1(a, b; c; x)$ is Gauss's usual hypergeometric function.

In analogy with Gauss' parameterization of the classical arithmetic-geometric mean in terms of theta functions, Borwein, Borwein, and Garvan obtained a modular parameterization [10, 11, 12] of the cubic AGM. To make this precise, let

$$x(z) := \frac{b(z)^3}{a(z)^3} = 1 - 27q + 405q^2 - 4617q^3 + 45333q^4 - \cdots, \tag{1.11}$$

where

$$a(z) := \sum_{m,n\in\mathbb{Z}} q^{n^2+mn+m^2} \quad \text{and} \quad b(z) := \sum_{m,n\in\mathbb{Z}} e^{\frac{2\pi i(n-m)}{3}} q^{n^2+nm+m^2}.$$

Their parameterization of (1.10) is captured by the identity

$${}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1-x(z)\right) = a(z) = 1 + 6q + 6q^3 + 6q^4 + 12q^7 + \cdots. \tag{1.12}$$

This q-series is itself a limit of a sequence. For suitable primes $p \equiv 2 \pmod 3$, it is one of two natural hypergeometric q-series which are p-adic limits of sequences of q-series obtained from harmonic Maass forms. Its companion is

$${}_2F_1\left(\frac{1}{3}, \frac{1}{3}; 1; 1-\frac{1}{x(z)}\right) = b(z) = 1 - 3q + 6q^3 - 3q^4 - 6q^7 + 6q^9 + \cdots. \tag{1.13}$$

To define these sequences, we let

$$F_1(x) := {}_2F_1\left(\frac{1}{3}, \frac{1}{3}; 1; 1-x\right) \quad \text{and} \quad F_2(x) := {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1-x\right) \tag{1.14}$$

and

$$F_3(x) := F_2(x) - F_1(1/x). \tag{1.15}$$

We then define $\Omega(z)$ by

(1.16)

$$\Omega(z) = \sum_{n=-1}^{\infty} C(n)q^n := \frac{F_1\left(1/x(z)\right) \cdot F_2\left(x(3z)\right)^2 \cdot F_3\left(x(z/3)\right)^3}{9F_3\left(x(z)\right)^2} = q^{-1} + 2q^2 - 49q^5 + \ldots.$$

Theorem 1.3. *If $\eta(z)$ is Dedekind's eta-function, then we have that*

$$\Omega(z) = \eta(3z)^8 \left(\frac{\eta(z)^3}{\eta(9z)^3} + 3\right)^2.$$

Moreover, if $p \equiv 2 \pmod 3$ *is a prime with* $p^3 \nmid C(p)$*, then as p-adic limits we have*

$$ {}_2F_1\left(\frac{1}{3},\frac{1}{3};1;1-\frac{1}{x(z)}\right) = \frac{729}{F_3(x(z/3))^3}\cdot \lim_{w\to+\infty} \frac{\Omega(z)\mid U(p^{2w+1})}{C(p^{2w+1})}, $$

$$ {}_2F_1\left(\frac{1}{3},\frac{2}{3};1;1-x(z)\right) = \frac{729a(z)}{b(z)F_3(x(z/3))^3}\cdot \lim_{w\to+\infty} \frac{\Omega(z)\mid U(p^{2w+1})}{C(p^{2w+1})}. $$

Remark 9. Since $C(n)=0$ for $n\equiv 0,1 \pmod 3$, Theorem 1.3 cannot hold for primes $p\equiv 1 \pmod 3$. This also explains the absence of even powers of p when $p\equiv 2 \pmod 3$.

Remark 10. We have checked that $p^3\nmid C(p)$ for every $p\equiv 2 \pmod 3$ less than 32500.

Example. We illustrate Theorem 1.3 when $p=2$. For convenience, we let

$$ \Omega_1(2w+1;z) := \frac{729}{F_3(x(z/3))^3}\cdot \frac{\Omega(z)\mid U(2^{2w+1})}{C(2^{2w+1})}. $$

Then we have that

$$ \Omega_1(1;z) = 1-3q+38q^3-99q^4-\cdots \equiv {}_2F_1\left(\frac{1}{3},\frac{1}{3};1;1-\frac{1}{x(z)}\right) \pmod{2^4}, $$

$$ \Omega_1(3;z) = 1-3q-\frac{9454}{3}q^3+9469q^4+\cdots \equiv {}_2F_1\left(\frac{1}{3},\frac{1}{3};1;1-\frac{1}{x(z)}\right) \pmod{2^7}, $$

$$ \Omega_1(5;z) = 1-3q-\frac{19856430}{1187}q^3+\cdots \equiv {}_2F_1\left(\frac{1}{3},\frac{1}{3};1;1-\frac{1}{x(z)}\right) \pmod{2^{10}}. $$

This illustrates the 2-adic convergence of $\Omega_1(2w+1;z)\longrightarrow {}_2F_1\left(\frac{1}{3},\frac{1}{3};1;1-\frac{1}{x(z)}\right)$.

This paper is organized as follows. In Section 2 we prove Theorems 1.1 and 1.2. We prove these theorems by extending earlier work [7] of Bruinier, Rhoades, and the third author, combined with a careful combinatorial analysis of the action of the Hecke operators $T(m)$. This analysis gives the desired implications for the properties of iterations of Atkin's $U(p)$ operator. In Section 3 we prove Theorem 1.3 using Theorem 1.2.

2. Proof of Theorems 1.1 and 1.2

Here we prove Theorems 1.1 and 1.2. We first prove Theorem 1.1 by refining earlier work of Bruinier, Rhoades and the third author [7].

2.1. **Proof of Theorem 1.1.** This result follows from a modification of the proof of Theorem 1.3 of [7]. Recall that we defined

$$ F_\alpha(z) := D^{k-1}\mathcal{F}_\alpha(z) = \sum_{n\gg-\infty} c_\alpha(n)q^n. $$

By construction, we have that $c_\alpha(1)$ is in K_g.

We use the action of the Hecke operators on $f(z)$. Let $T(m)$ be the mth Hecke operator for the group $\Gamma_0(N)$ and Nebentypus χ. Using the same argument as in Lemma 7.4 of [13] (cf. [7], the proof of Theorem 1.3), we have that

$$ f|_{2-k}T(m) = m^{1-k}b_g(m)f + R_m, \tag{2.1} $$

where $R_m \in M^!_{2-k}(\Gamma_0(N), \chi)$ is a weakly holomorphic modular form with coefficients in K_g. The point is that $f|_{2-k}T(m)$ and $m^{1-k}b_g(m)f$ are harmonic Maass forms with equal non-holomorphic part. We apply the differential operator D^{k-1} to this identity, and we use the commutation relation

$$m^{k-1}D^{k-1}\left(H|_{2-k}T(m)\right) = \left(D^{k-1}H\right)\ |_k\ T(m)$$

which is valid for any 1-periodic function H. We obtain

$$\left(D^{k-1}f\right)|_k T(m) = b_g(m)\left(D^{k-1}f\right) + m^{k-1}D^{k-1}R_m.$$

Since $D^{k-1}f = D^{k-1}f^+$, and $D^{k-1}E_g = g$, and since $g\ |_k\ T(m) = b_g(m)g$, we conclude that

$$\left(D^{k-1}(f^+ - \alpha E_g)\right)|_k T(m) = b_g(m)\left(D^{k-1}(f^+ - \alpha E_g)\right) + m^{k-1}D^{k-1}R_m. \tag{2.2}$$

We claim that the q-series $F_\alpha = D^{k-1}(f^+ - \alpha E_g) = D^{k-1}\mathcal{F}_\alpha$ has its coefficients in K_g. Indeed, we make use of the formula for the action of Hecke operators on Fourier expansions, equate the coefficients of q^n in (2.2), and conclude that for any prime m

$$c_\alpha(mn) + \chi(m)m^{k-1}c_\alpha(n/m) - b_g(m)c_\alpha(n) \in K_g.$$

An inductive argument, using the fact that $c_\alpha(1)$ is in K_g, finishes the proof.

2.2. **Proof of Theorem 1.2.** We assume the notation and hypotheses from the introduction. We require the following elementary proposition.

Proposition 2.1. *If $R \in M^!_{2-k}(\Gamma_0(N), \chi)$ has K_g-coefficients, then there is an A such that*

$$\mathrm{ord}_p((D^{k-1}R)\mid U(p^n)) \geq n(k-1) - A.$$

Proof. The coefficients $a(n)$ of R have bounded denominators. In other words, we have that $A := -\inf_n\left(\mathrm{ord}_p(a(n))\right) < \infty$. Indeed, we can always multiply R by an appropriate power of Δ, and obtain a cusp form of positive integer weight, which has Fourier coefficients with bounded denominators as a linear combination of forms with rational integral Fourier coefficients by Theorem 3.52 of [14]. Dividing back by the power of Δ preserves this property since the coefficients of $1/\Delta$ are integers. The proposition now follows easily from

$$(D^{k-1}R)\mid U(p^n) = \sum_{m \gg -\infty}(p^n m)^{k-1}a(p^n m)q^m.$$

□

We now prove the existence of the limits which appear in Theorem 1.2.

Proposition 2.2. *Assuming the hypotheses in Theorem 1.2 (1), we have that*

$$\lim_{w\to\infty}\beta^{-w}F_\alpha \mid U(p^w) \in \mathbb{C}_p[[q]].$$

Proof. We assume that $p \nmid N$. The proof when $p \mid N$ is similar.

Recall that the weight k Hecke operator $T(p)$ acts by

$$F_\alpha(z)\mid T(p) = F_\alpha(z)\mid U(p) + \chi(p)p^{k-1}F_\alpha(pz).$$

Then (2.2) with $m = p$ gives

$$F_\alpha(z)\mid U(p) + \beta\beta' F_\alpha(pz) = (\beta + \beta')F_\alpha(z) + r,$$

where $r := p^{k-1}D^{k-1}R_p$ is a weakly holomorphic modular form in $M_k^!(\Gamma_0(N),\chi)$. We made use of (1.8), which implies that

$$\beta+\beta' = b_g(p) \quad \text{and} \quad \beta\beta' = \chi(p)p^{k-1}.$$

Now we let

$$G(z) := F_\alpha(z) - \beta' F_\alpha(pz) \ \text{ and } \ G'(z) := F_\alpha(z) - \beta F_\alpha(pz). \tag{2.3}$$

A simple calculation reveals that

$$G \mid U(p) = \beta G + r \quad \text{ and } \quad G' \mid U(p) = \beta' G' + r,$$

and also that

$$F_\alpha \mid U(p) = \frac{\beta}{\beta-\beta'}(\beta G + r) - \frac{\beta'}{\beta-\beta'}(\beta' G' + r).$$

By induction, we find that

$$\begin{aligned}(\beta-\beta')\beta^{-w}F_\alpha \mid U(p^w) = \ & \left(\beta G + r + \tfrac{1}{\beta} r \mid U(p) + \ldots + \tfrac{1}{\beta^{w-1}} r \mid U(p^{w-1})\right) \\ & -(\beta'/\beta)^w \left(\beta' G' + r + \tfrac{1}{\beta'} r \mid U(p) + \ldots + \tfrac{1}{\beta'^{w-1}} r | U(p^{w-1})\right).\end{aligned}$$

Proposition 2.2 now follows from Proposition 2.1 and this formula. □

Now we prove that the limits in Theorem 1.2 (2) are well defined.

Proposition 2.3. *Assuming the hypotheses in Theorem 1.2 (2), we have that*

$$\lim_{w\to\infty} \beta^{-2w} G_\alpha \mid U(p^{2w+1}) \in \mathbb{C}_p[[q]].$$

Proof. Observe that

$$\lim_{w\to\infty} \beta^{-2w} E_g \mid U(p^{2w}) \in \mathbb{C}_p[[q]].$$

It thus suffices to check the statement of the proposition for $\alpha = 0$. Since p is inert in the CM-field, we have $\beta' = -\beta$, and so $\beta^2 = -\chi(p)p^{k-1}$. As in the proof of Proposition 2.2, we rewrite equation (2.2) with $m = p$:

$$G_0(z) \mid U(p) - \beta^2 G_0(pz) = r,$$

where $r := p^{k-1}D^{k-1}R_p$ is a weakly holomorphic modular form in $M_k^!(\Gamma_0(N),\chi)$. Thus we have that

$$G_0 \mid U(p^2) = \beta^2 G_0 + r \mid U(p).$$

Acting with the U-operator on this identity $2w-1$ times, we obtain

$$\beta^{-2w} G_0 \mid U(p^{2w+1}) = G_0 \mid U(p) + \beta^{-2} r \mid U(p^2) + \beta^{-4} r \mid U(p^4) + \ldots + \beta^{-2w} r \mid U(p^{2w}). \tag{2.4}$$

As in Proposition 2.2, we conclude that the p-adic limit exists. □

Proof of Theorem 1.2. Here we prove Theorem 1.2 (1). The proof of Theorem 1.2 (2) is similar apart from the fact that one applies Proposition 2.3 in place of Proposition 2.2.

We begin by considering the first Fourier coefficient in Proposition 2.2, and put

$$L = L(\alpha) := \lim_{w\to\infty} \beta^{-w} c_\alpha(p^w).$$

Since $\lim_{w\to\infty} \beta^{-w} b_g(p^w) = \beta/(\beta-\beta')$, and $c_\alpha(n) = n^{k-1}c_f(n) - \alpha b_g(n)$, there is at most one choice of $\alpha \in K_g$ for which $L(\alpha) = 0$. For non-exceptional α, we can then conclude that $c_\alpha(p^n) \neq 0$ for $n \gg 0$.

Let

$$L^{-1} \lim_{w\to\infty} \beta^{-w} F_\alpha \mid U(p^w) = \sum_{m>0} \check{c}(m) q^m. \tag{2.5}$$

It follows from the proof of Proposition 2.2 that

$$\left(\lim_{w\to\infty} \beta^{-w} F_\alpha \mid U(p^w)\right) \mid U(p) = \beta \left(\lim_{w\to\infty} \beta^{-w} F_\alpha \mid U(p^w)\right).$$

Therefore, by (1.9), (2.5), and the recursive formula for $b_g(p^n)$, we inductively find that

$$\check{c}(p^n) = \beta^n = \check{b}_g(p^n)$$

for all $n \geq 0$. For $m > 0$ such that $p \nmid m$, we then have that

$$F_\alpha \mid T(m) = \check{b}_g(m) F_\alpha + r_m$$

with $r_m = m^{k-1} D^{k-1} R_m$. Since the operators $U(p)$ and $T(m)$ commute, we obtain

$$(F_\alpha \mid U(p^w)) \mid T(m) = \check{b}_g(m) (F_\alpha \mid U(p^w)) + r_m \mid U(p^w).$$

We divide this equation by $c_\alpha(p^w)$, and then take the limit as $w \to +\infty$. By Proposition 2.1, the formulas for Hecke operators, and the property that Fourier coefficients of Hecke eigenforms are the eigenvalues, Theorem 1.2 (1) follows easily. ☐

3. p-adic properties of the cubic AGM

Here we prove Theorem 1.3 by combining properties of a convenient family of modular forms w_l with known results concerning the modular forms which arise in the cubic AGM.

3.1. **A family of convenient modular forms.** Here we construct a family of modular forms which are vital to the proof of Theorem 1.3.

Proposition 3.1. *If $l \geq 2$ is an integer, then there is a $w_l \in M^!_{-2}(\Gamma_0(9))$, which is bounded at all cusps of $\Gamma_0(9)$ apart from infinity, and a $t_l \in \mathbb{C}$ for which*

$$w_l - q^{-l} - t_l q^{-1} \in \mathbb{Z}[[q]]. \tag{3.1}$$

Proof. To prove the existence of w_l, it suffices to construct functions W_l for $l \geq 2$ which satisfy all conditions of Proposition 3.1 with the weaker condition that

$$W_l - q^{-l} \in q^{-l+1}\mathbb{Z}[[q]].$$

This is clear since one may diagonalize iteratively to manufacture the desired w_l.

Using Theorem 1.65 of [15], one finds $w_2 := \frac{\eta(3z)^2}{\eta(9z)^6}$. For integers $s \geq 2$ we have weakly holomorphic modular forms $w_2^s \in M^!_{-2s}(\Gamma_0(9))$ with q-expansions satisfying

$$w_2^s - q^{-2s} \in q^{-2s+1}\mathbb{Z}[[q]].$$

Note that the functions w_2^s are still bounded at all cusps except infinity. We now claim the existence of holomorphic modular forms $\varphi_s, \psi_s \in M_{2s-2}(\Gamma_0(9)) \cap \mathbb{Z}[[q]]$ such that

$$\varphi_s \in q\mathbb{Z}[[q]] \quad \text{and} \quad \psi_s - 1 \in q\mathbb{Z}[[q]].$$

Then we let

$$W_l := \begin{cases} w_2^s \varphi_s & \text{if} \quad l = 2s-1 \\ w_2^s \psi_s & \text{if} \quad l = 2s. \end{cases}$$

To complete the proof, we must exhibit the φ_s and ψ_s. Let $E_4 = 1+240\sum_{n\geq 1}\sigma_3(n)q^n$ and $E_6 = 1-504\sum_{n\geq 1}\sigma_5(n)q^n$ be the standard Eisenstein series of weights 4 and 6 respectively. Then for all $s \geq 3$ we may set $\psi_s := E_4^a E_6^b$ with non-negative integers a and b such that $4a+6b = 2s-2$. For $s=2$ we let $\psi_2 := \eta(z)^6/\eta(3z)^2$. The cusp forms $\varphi_s := \Delta E_4^c E_6^d$, with $4c+6d+12 = 2s-2$, do the job for $s=7$ and all $s\geq 9$. For $s=2$ we put

$$\varphi_2 := (\eta(3z)^{10}/(\eta(z)^3\eta(9z)^3) - \eta(z)^6/\eta(3z)^2)/9 = q+7q^4+8q^7+18q^{10}+14q^{13}+31q^{16}+\dots.$$

It is not difficult to check that this form has integer coefficients. Using Theorem 1.65 [15] again, one easily checks that φ_2 is bounded at all cusps. We put

$$\varphi_s = \begin{cases} \eta(3z)^8 & \text{if } \ s=3 \\ \eta(3z)^8\psi_{s-2} & \text{if } \ s=4,5,6,8. \end{cases}$$

□

3.2. **Modular form identities.** We also require a number of identities relating the modular forms $a(z)$ and $b(z)$. To this end we also require the q-series $c(z)$ defined by

$$c(z) := \sum_{m,n\in\mathbb{Z}} q^{\left(n+\frac{1}{3}\right)^2+\left(n+\frac{1}{3}\right)\left(m+\frac{1}{3}\right)+\left(m+\frac{1}{3}\right)^2} = 3q^{\frac{1}{3}}+3q^{\frac{4}{3}}+6q^{\frac{7}{3}}+6q^{\frac{13}{3}}+\cdots. \tag{3.2}$$

Theorem 3.2. *Assuming the notation above, the following identities are true:*

$$c(3z) = \frac{1}{3}\left(a(z)-b(z)\right), \tag{3.3}$$

$$\eta(3z)^8 = \frac{1}{27}\cdot b(z)c(z)^3, \tag{3.4}$$

$$\frac{\eta(z)^3}{\eta(9z)^3}+3 = \frac{3a(3z)}{c(3z)}. \tag{3.5}$$

Proof. These identities follow from work by Garvan (see pages 250-256 of [12]). Identities (1.12) and (1.13) are identities (2.16) on page 252 and (2.37) on page 255 respectively. Identity (3.3) is (2.27) on page 253.

Formula (3.4) follows from the identities

$$b(z) = \prod_{n=1}^{\infty}\frac{(1-q^n)^3}{(1-q^{3n})} \qquad \text{and} \qquad c(z) = 3q^{\frac{1}{3}}\prod_{n=1}^{\infty}\frac{(1-q^{3n})^3}{(1-q^n)} \tag{3.6}$$

which are (2.24) and (2.25) on page 253 of [12]. To prove (3.5), we employ (3.6) to find that

$$\frac{3a(3z)}{c(3z)} = \frac{\eta(3z)a(3z)}{\eta(9z)^3}.$$

Formula (2.33) on page 255 of [12] asserts that $b(z) = a(3z)-c(3z)$. Solving for $a(3z)$, one obtains (3.5) thanks to (3.6) above. □

3.3. **Proof of Theorem 1.3.** We first prove the claim that

$$\Omega(z) = \sum_{n=-1}^{\infty} C(n)q^n = \eta(3z)^8 \cdot \left(\frac{\eta(z)^3}{\eta(9z)^3} + 3\right)^2.$$

By (3.4) and (3.5), we have that

$$\eta(3z)^8 \cdot \left(\frac{\eta(z)^3}{\eta(9z)^3} + 3\right)^2 = \frac{a(3z)^2 b(z) c(z)^3}{3c(3z)^2}.$$

The claimed identity for $\Omega(z)$ now follows from (1.12), (1.13) and (3.3).

It is well known that $g(z) = \eta(3z)^8$ is the unique normalized newform in $S_4(\Gamma_0(9))$. Moreover, it is a modular form with complex multiplication with respect to $\mathbb{Q}(\sqrt{-3})$. In a recent paper [7], Bruinier, Rhoades and the third author produced a weight -2 harmonic Maass form, which we call f, which is good for $g = g^c$ (see Section 7 of [7]). Moreover, the image of this form under the differential operator D^3 turns out to be $-\Omega(z)$. One then easily finds that the first few coefficients of the regularized mock modular form f^+ are

$$\mathcal{F}_0 = f^+ = q^{-1} - \frac{1}{4}q^2 + \frac{49}{125}q^5 - \dots. \tag{3.7}$$

Theorem 1.2 (2) then implies, for suitable primes $p \equiv 2 \pmod 3$, that

$$\lim_{w \to +\infty} \frac{\Omega(z) \mid U(p^{2w+1})}{C(p^{2w+1})} = g(z) = \eta(3z)^8.$$

Below we shall prove that this conclusion holds for any prime $p \equiv 2 \pmod 3$ for which $p^3 \nmid C(p)$. This condition guarantees that $\alpha = 0$ is not exceptional. When this is the case, (3.4) then implies that

$$b(z) = \frac{27}{c(z)^3} \cdot \lim_{w \to +\infty} \frac{\Omega(z) \mid U(p^{2w+1})}{C(p^{2w+1})}.$$

The claim that

$${}_2F_1\left(\frac{1}{3}, \frac{1}{3}; 1; 1 - \frac{1}{x(z)}\right) = \frac{729}{F_3(x(z/3))^3} \cdot \lim_{w \to +\infty} \frac{\Omega(z) \mid U(p^{2w+1})}{C(p^{2w+1})}$$

now follows from (1.12), (1.13), and (3.3). The limit for ${}_2F_1\left(\frac{1}{3}, \frac{2}{3}, 1; 1 - x(z)\right)$ follows immediately from (1.12), (1.13) and the trivial observation that

$${}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1 - x(z)\right) = a(z) = \frac{a(z)}{b(z)} \cdot b(z) = \frac{a(z)}{b(z)} \cdot {}_2F_1\left(\frac{1}{3}, \frac{1}{3}; 1; 1 - \frac{1}{x(z)}\right). \tag{3.8}$$

To complete the proof, it suffices to show that $\alpha = 0$ is not exceptional for a prime $p \equiv 2 \pmod 3$ when $p^3 \nmid C(p)$. Suppose that p is such a prime. It then suffices to show that

$$\lim_{w \to \infty} \beta^{-2w} \Omega \mid U(p^{2w+1}) \neq 0 \tag{3.9}$$

On the contrary, if this limit is 0, then (2.4) implies that

$$-\Omega \mid U(p) = \sum_{w \geq 1} \beta^{-2w} r \mid U(p^{2w}). \tag{3.10}$$

Recall that $r = p^3 D^3 R_p$, and let

$$p^3 R_p = \sum_{n \gg -\infty} a(n) q^n. \tag{3.11}$$

We equate the coefficients of q in (3.10), and obtain (taking into account that $k = 4$)

$$-C(p) = \sum_{w \geq 1} (-1)^w p^{3w} a(p^{2w}).$$

Obviously, the truth of (3.9), under the assumption that $p^3 \nmid C(p)$, follows if the numbers $a(n)$ are integers for $n \geq 1$.

We complete the proof of the theorem by establishing that these numbers are indeed integers. Since $p \equiv 2 \pmod 3$, it follows that $b_g(p) = 0$. Therefore, (2.1) implies that

$$f \mid_{-2} T(p) = f^+ \mid_{-2} T(p) = R_p,$$

where R_p is in $M^!_{-2}(\Gamma_0(9))$. Thanks to (3.7), we then see that

$$p^3 R_p = q^{-p} + \sum_{n=1}^{\infty} a(n) q^n.$$

Using the weakly holomorphic modular form w_p, we then find by construction that

$$\widetilde{\Omega}_p := -w_p + p^3 R_p + t_p(f^+ + f^-)$$

is a harmonic Maass form whose holomorphic parts are bounded at all cusps. This form has the additional property that $\xi(\widetilde{\Omega}_p) = t_p g / \langle g, g \rangle$. By Proposition 3.5 [2], it is known that a harmonic Maass form which is not a weakly holomorphic modular form has a nonconstant principal part at some cusp. Therefore, it follows that $t_p = 0$. Consequently, we have that $-w_p + p^3 R_p$ is a weight -2 holomorphic modular form. The only such form is 0, and so we have that $w_p = p^3 R_p$. The integrality of the $a(n)$ now follows from Proposition 3.1, and the proof of Theorem 1.3 is complete.

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