“Easy” Session

To lighten the mood just before the big day, today we’ll look at some “easier” Putnam questions, from 2010 and earlier. These are often (but not always) problems A1 and B1. There is no fixed mathematical theme here.

This handout has my usual format: problems on the first page, hints on the second page, and solutions on the remaining pages.

1. PROBLEMS

Look over these problems. Find one (or more) where you have some idea of how to get started. I recommend trying this for at least a half an hour, before looking at the next page!

2010 A1: Given a positive integer $n$, what is the largest $k$ such that the numbers $1, 2, \ldots, n$ can be put into $k$ boxes so that the sum of the numbers in each box is the same? [When $n = 8$, the example \{1, 2, 3, 6\}, \{4, 8\}, \{5, 7\} shows that the largest $k$ is at least 3.]

2010 B1: Is there an infinite sequence of real numbers $a_1, a_2, a_3, \ldots$ such that

$$a_1^m + a_2^m + a_3^m + \cdots = m$$

for every positive integer $m$?

2009 A1: Let $f$ be a real-valued function on the plane such that for every square $ABCD$ in the plane, $f(A) + f(B) + f(C) + f(D) = 0$. Does it follow that $f(P) = 0$ for all points $P$ in the plane?

2009 B1: Show that every positive rational number can be written as a quotient of products of factorials of (not necessarily distinct) primes. For example,

$$\frac{10}{9} = \frac{2! \cdot 5!}{3! \cdot 3! \cdot 3!}.$$

2008 A1: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that $f(x,y) + f(y,z) + f(z,x) = 0$ for all real numbers $x, y, z$. Prove that there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = g(x) - g(y)$ for all real numbers $x$ and $y$.

2008 B1: What is the maximum number of rational points that can lie on a circle in $\mathbb{R}^2$ whose center is not a rational point? (A rational point is a point both of whose coordinates are rational numbers.)

2007 A2: Find the least possible area of a convex set in the plane that intersects both branches of the hyperbola $xy = 1$ and both branches of the hyperbola $xy = -1$. (A set $S$ in the plane is called convex if for any two points in $S$ the line segment connecting them is contained in $S$.)

2007 B1: Let $f$ be a non-constant polynomial with positive integer coefficients. Prove that if $n$ is a positive integer, then $f(n)$ divides $f(f(n) + 1)$ if and only if $n = 1$. 

1
II. HINTS

You don’t get hints on a real exam, but hints like these may help you with similar problems. Look these over, and see if you can make any further progress before looking at the solutions.

2010 A1: Equivalently, how small can you make the common sum?

2010 B1: Try the Cauchy-Schwarz inequality (for vectors: $|u \cdot v| \leq |u| \cdot |v|$).

2009 A1: Consider squares, smaller squares, and rotated squares, all around $P$.

2009 B1: Try induction on the largest prime involved.

2008 A1: Try $z = 0$, or $y = z = 0$, to find some nice properties of $f$.

2008 B1: Argue two rational points determine a “rational line”, and vice-versa.

2007 A2: Quadrilateral. Area of triangle as a determinant. The arithmetic mean of $n$ numbers is at least their geometric mean.

2007 B1: Compute $f(f(n) + 1) \mod f(n)$.

The next page has solutions, don’t continue until you really want to see them!
III. SOLUTIONS

2010 A1: The largest such $k$ is $\left\lfloor \frac{n+1}{2} \right\rfloor = \left\lceil \frac{n}{2} \right\rceil$. For $n$ even, this value is achieved by the partition

$$\{1,n\},\{2,n-1\},\ldots$$

for $n$ odd, it is achieved by the partition

$$\{n\},\{1,n-1\},\{2,n-2\},\ldots$$

One way to see that this is optimal is to note that the common sum can never be less than $n$, since $n$ itself belongs to one of the boxes. This implies that $k \leq (1 + \cdots + n)/n = (n+1)/2$. Another argument is that if $k > (n+1)/2$, then there would have to be two boxes with one number each (by the pigeonhole principle), but such boxes could not have the same sum.

2010 B1: No such sequence exists. If it did, the Cauchy-Schwartz inequality would imply

$$8 = (a_1^2 + a_2^2 + \cdots)(a_1^4 + a_2^4 + \cdots) \geq (a_1^2 + a_2^2 + \cdots)^2 = 9,$$

contradiction. (A second solution is omitted here.)

2009 A1: Yes, it does follow. Let $P$ be any point in the plane. Let $ABCD$ be any square with center $P$. Let $E,F,G,H$ be the midpoints of the segments $AB,BC,CD,DA$, respectively. The function $f$ must satisfy the equations

$$0 = f(A) + f(B) + f(C) + f(D),$$
$$0 = f(E) + f(F) + f(G) + f(H),$$
$$0 = f(A) + f(E) + f(P) + f(H),$$
$$0 = f(B) + f(F) + f(P) + f(E),$$
$$0 = f(C) + f(G) + f(P) + f(F),$$
$$0 = f(D) + f(H) + f(P) + f(G).$$

If we add the last four equations, then subtract the first and twice the second, we obtain $0 = 4f(P)$.

2009 B1: Every positive rational number can be uniquely written in lowest terms as $a/b$ for $a, b$ positive integers. We prove the statement in the problem by induction on the largest prime dividing either $a$ or $b$ (where this is considered to be 1 if $a = b = 1$). For the base case, we can write $1/1 = 2!/2!$. For a general $a/b$, let $p$ be the largest prime dividing either $a$ or $b$; then $a/b = p^ka'/b'$ for some $k \neq 0$ and positive integers $a', b'$ whose largest prime factors are strictly less than $p$. We now have $a/b = (p^k)^{\frac{a'}{(p-1)!b'}}$, and all prime factors of $a'$ and $(p-1)!b'$ are strictly less than $p$. By the induction assumption, $\frac{a'}{(p-1)!b'}$ can be written as a quotient of products of prime factorials, and so $a/b = (p^k)^{\frac{a'}{(p-1)!b'}}$ can as well. This completes the induction.

2008 A1: The function $g(x) = f(x,0)$ works. Substituting $(x,y,z) = (0,0,0)$ into the given functional equation yields $f(0,0) = 0$, whence substituting $(x,y,z) = (x,0,0)$ yields $f(x,0) + f(0,x) = 0$. Finally, substituting $(x,y,z) = (x,y,0)$ yields $f(x,y) = -f(y,0) - f(0,x) = g(x) - g(y)$.

Remark: A similar argument shows that the possible functions $g$ are precisely those of the form $f(x,0) + c$ for some $c$.

2008 B1: There are at most two such points. For example, the points $(0,0)$ and $(1,0)$ lie on a circle with center $(1/2,x)$ for any real number $x$, not necessarily rational.

On the other hand, suppose $P = (a,b), Q = (c,d), R = (e,f)$ are three rational points that lie on a circle. The midpoint $M$ of the side $PQ$ is $((a+c)/2, (b+d)/2)$, which is again rational. Moreover, the slope of the line $PQ$ is $(d-b)/(c-a)$, so the slope of the line through $M$ perpendicular to $PQ$ is $(a-c)/(b-d)$, which is rational or infinite. Similarly, if $N$ is the midpoint of $QR$, then $N$ is a rational point and the line through $N$ perpendicular to $QR$ has rational slope. The center of the circle lies on both of these lines, so its coordinates $(g,h)$ satisfy two linear equations with rational coefficients, say $Ag + Bh = C$ and $Dg + Eh = F$. Moreover, these equations have a unique solution. That solution must then be

$$g = (CE - BD)/(AE - BD)$$
$$h = (AF - BC)/(AE - BD)$$
(by elementary algebra, or Cramer’s rule), so the center of the circle is rational. This proves the desired result.

**Remark:** The above solution is deliberately more verbose than is really necessary. A shorter way to say this is that any two distinct rational points determine a *rational line* (a line of the form $ax + by + c = 0$ with $a, b, c$ rational), while any two nonparallel rational lines intersect at a rational point.

**2007 A2:** The minimum is 4, achieved by the square with vertices $(±1, ±1)$.

To prove that 4 is a lower bound, let $S$ be a convex set of the desired form. Choose $A, B, C, D ∈ S$ lying on the branches of the two hyperbolas, with $A$ in the upper right quadrant, $B$ in the upper left, $C$ in the lower left, $D$ in the lower right. Then the area of the quadrilateral $ABCD$ is a lower bound for the area of $S$.

Write $A = (a, 1/a), B = (b, −1/b), C = (−c, −1/c), D = (−d, 1/d)$ with $a, b, c, d > 0$. Then the area of the quadrilateral $ABCD$ is

$$\frac{1}{2}(a/b + b/c + c/d + d/a + b/a + c/b + d/c + a/d),$$

which by the arithmetic-geometric mean inequality is at least 4.

**2007 B1:** Write $f(n) = \sum_{i=0}^{d} a_i n^i$ with $a_i > 0$. Then

$$f(f(n) + 1) = \sum_{i=0}^{d} a_i (f(n) + 1)^i \equiv f(1) \pmod{f(n)}.$$ 

If $n = 1$, then this implies that $f(f(n) + 1)$ is divisible by $f(n)$. Otherwise, $0 < f(1) < f(n)$ since $f$ is nonconstant and has positive coefficients, so $f(f(n) + 1)$ cannot be divisible by $f(n)$.