MATH 385 - PROBLEM SOLVING-SOLVING STRATEGIES - FALL 2025

HOMEWORK ASSIGNMENT 1
DUE: 11:59 PM, XXXXXXX, SEPTEMBER X

Problem 1. Show that for all positive integers n,

$$\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$$

is an integer.

Problem 2. Prove that for all positive integers n the identity

$$\frac{1}{n+1} + \cdots + \frac{1}{2n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}.$$

Problem 3. Prove that $3^n \geq n^3$ for all positive integers n.

Problem 4. Let $x_1, x_2, \ldots, x_n$, $y_1, y_2, \ldots, y_m$ be positive integers, with $n, m > 1$. Assume that

$$x_1 + x_2 + \cdots + x_n \ = \ y_1 + y_2 + \cdots + y_m \ < \ mn.$$

Prove that in the equality

$$x_1 + x_2 + \cdots + x_n \ = \ y_1 + y_2 + \cdots + y_m$$

one can remove some (but not all) terms in such a way that the equality is still satisfied.

Problem 5. Prove that any positive integer can be represented as

$$\pm 1^2 \ \pm 2^2 \ \pm \cdots \ \pm n^2$$

for some positive integer n and some choice of the signs.

Problem 6 (Putnam 2003, A1)**.** Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers

$$n = a_1 + a_2 + \cdots + a_k,$$

where k is an arbitrary positive integer and

$$a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1\,?$$

For example, when $n = 4$, there are four such representations:

$$4, \quad 2+2, \quad 1+1+2, \quad 1+1+1+1.$$

Problem 7 (Putman 2005, A1)**.** Show that every positive integer can be represented as a sum of one or more numbers of the form $2^r 3^s$, where $r, s \geq 0$ are integers, such that no summand divides another.

Problem 8 (Mantel's theorem)**.** Take $2n$ points in space and connect some pairs of them with line segments. If the total number of segments is at least $n^2 + 1$, then there will always be three of the points such that each pair of them is connected by a segment, that is, they form a triangle.

Problem 9 (Dvoretzky–Motzkin, Raney)**.** Let $(x_1, \ldots, x_n)$ be real numbers indexed cyclically by $\mathbb{Z}_n$ with

$$\sum_{i=1}^{n} x_i = 0.$$

Then there exists $i \in \mathbb{Z}_n$ such that all cyclic partial sums

$$x_i + \cdots + x_{i+k}$$

are positive for $k = 0, 1, \ldots, n-2$, and the sum for $k = n-1$ equals 0.

Problem 10 (Putnam, 1995, A4)**.** Suppose we have a necklace of n beads. Each bead is labeled with an integer, and the sum of all these labels is $n-1$. Prove that we can cut the necklace to form a string whose consecutive labels $x_1, \ldots, x_n$ satisfy

$$\sum_{i=1}^{k} x_i \leq k-1 \quad \text{for all } k = 1, 2, \ldots, n.$$