Matrix Session

Today we look at a few old Putnam questions involving matrices.
As usual: first page has questions, second page has hints, remaining pages have solutions.

I. PROBLEMS

Look over the problems below. Try to identify one or more problems where you have some idea of how to get started. For a real Putnam session, I recommend you spend at least half an hour just on this step!

1992 B5: Let $D_n$ denote the value of the $(n - 1) \times (n - 1)$ determinant

\[
\begin{pmatrix}
3 & 1 & 1 & \cdots & 1 \\
1 & 4 & 1 & \cdots & 1 \\
1 & 1 & 5 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & \cdots & n + 1
\end{pmatrix}.
\]

Is the set $\{ \frac{D_n}{n!} \}_{n \geq 2}$ bounded?

1990 A5: If $A$ and $B$ are square matrices of the same size such that $ABAB = 0$, does it follow that $BABA = 0$?

2016 B4: Let $A$ be a $2n \times 2n$ matrix, with entries chosen independently at random. Every entry is chosen to be 0 or 1, each with probability $1/2$. Find the expected value of $\det(A - A')$ (as a function of $n$), where $A'$ is the transpose of $A$. 

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II. HINTS

You won’t get hints on a real exam, but these ideas that may help you with similar problems, might help. Look these over, and see if you can make any further progress.

1992 B5: Try subtracting one row or column from all the others, to simplify.

1990 A5: Look for a counterexample, think about paths in graphs.

2016 B4: A determinant is a sum over permutations, and a permutation is a product of orbits. Show that only orbits of size two can contribute.

The next page has solutions, don’t continue until you want to see them!
III. SOLUTIONS

1992 B5: No. Subtract column 1 from all the others. So the diagonal entries become 3, 3, 4, …, n. The first row has -2 apart from the first item, the first column remains 1 apart from the first item, and all other elements are 0. Subtract 1/3 of column 2 from column 1, 1/4 of col 3 from col 1, 1/5 of col 4 from col 1 and so on. This zeros all elements of col 1 except the first which becomes 3 + 2(1/3 + 1/4 + … + 1/n). We may now expand by the first column to get n!/2 times 3 + 2(1/3 + 1/4 + … + 1/n). So \((\text{det } A_n)/n! = (3 + 2(1/3 + 1/4 + … + 1/n))/2\). But this diverges as \(n\) tends to infinity.

1990 A5: No, here is a counterexample with 3 by 3 matrices. Suppose \(A_{12} = A_{31} = B_{11} = B_{33} = 1\), all others entries 0. Then \((AB)_{31} = 1\) and all other entries of \(AB\) are zero, so \(ABAB = 0\). On the other hand \((BA)_{31} = (BA)_{12} = 1\) and all other entries of \(BA\) are zero, so \(BABA\) is not zero (its 3,2 entry is 1).

2016 B4: The expected value is \((2n)!/(4^n n!)\).

The determinant of \(A - A'\) is the sum over permutations \(\sigma\) of \(\{1, \ldots, 2n\}\) of the product \(\text{sgn}(\sigma) \prod_{i=1}^{2n} (A - A')_{\sigma(i)} = \text{sgn}(\sigma) \prod_{i=1}^{2n} (A_{\sigma(i)} - A_{\sigma(i)')})\). The expected value of the determinant is the sum over \(\sigma\) of the expected value of this product, which we denote \(E_\sigma\).

Note that if we partition \(\{1, \ldots, 2n\}\) into orbits for the action of \(\sigma\), then partition the factors of the product accordingly, then no entry of \(A\) appears in more than one of these factors; consequently, these factors are independent random variables. This means that we can compute \(E_\sigma\) as the product of the expected values of the individual factors.

Any orbit of size 1 gives rise to the zero product, and hence the expected value of the corresponding factor is zero. For an orbit of size \(m \geq 3\), the corresponding factor contains \(2m\) distinct matrix entries, so again we may compute the expected value of the factor as the product of the expected values of the individual terms \(A_{\sigma(i)} - A_{\sigma(i)'}\). However, the distribution of this term is symmetric about 0, so its expected value is 0.

We conclude that \(E_\sigma = 0\) unless \(\sigma\) acts with \(n\) orbits of size 2. To compute \(E_\sigma\) in this case, assume without loss of generality that the orbits of \(\sigma\) are \(\{1, 2\}, \ldots, \{(2n-1), 2n\}\); note that \(\text{sgn}(\sigma) = (-1)^n\). Then \(E_\sigma\) is the expected value of \(\prod_{i=1}^{n} -{(A_{12} - A_{21})^2} = -{(A_{12} - A_{21})^2}^n\), which is \((-1)^n\) times the \(n\)-th power of the expected value of \((A_{12} - A_{21})^2\). Since \(A_{12} - A_{21}\) takes the values \(-1, 0, 1\) with probabilities \(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\), its square takes the values 0, 1 with probabilities \(\frac{1}{2}, \frac{1}{2}\); we conclude that \(E_\sigma = 2^{-n}\).

The permutations \(\sigma\) of this form correspond to unordered partitions of \(\{1, \ldots, 2n\}\) into \(n\) sets of size 2, so there are \((2n)!/(n!(2!)^n)\) such permutations. Putting this together yields the result.