Polynomial Putnam Problems

Below are some Putnam problems using basic properties of polynomials. Problems are on the first page, hints are on the second page, and solutions are after that.

1 Problems

Look over the problems below. Try to identify one or more problems where you have some idea of how to start. If you make any progress, try to write it down!

2017 A–2 Let $Q_0(x) = 1$, $Q_1(x) = x$, and

$$Q_n(x) = \frac{(Q_{n-1}(x))^2 - 1}{Q_{n-2}(x)}$$

for all $n \geq 2$. Show that, whenever n is a positive integer, $Q_n(x)$ is equal to a polynomial with integer coefficients.

2013 A–3 Suppose that the real numbers $a_0, a_1, \ldots, a_n$ and x, with $0 < x < 1$, satisfy

$$\frac{a_0}{1-x} + \frac{a_1}{1-x^2} + \cdots + \frac{a_n}{1-x^{n+1}} = 0.$$

Prove that there exists a real number y with $0 < y < 1$ such that

$$a_0 + a_1 y + \cdots + a_n y^n = 0.$$

2010 B–4 Find all pairs of polynomials $p(x)$ and $q(x)$ with real coefficients for which

$$p(x)q(x+1) - p(x+1)q(x) = 1.$$

2008 A–5 Let $n \geq 3$ be an integer. Let $f(x)$ and $g(x)$ be polynomials with real coefficients such that the points $(f(1), g(1)), (f(2), g(2)), \ldots, (f(n), g(n))$ in $\mathbb{R}^2$ are the vertices of a regular n-gon in counterclockwise order. Prove that at least one of $f(x)$ and $g(x)$ has degree greater than or equal to $n-1$.

2007 B–1 Let f be a polynomial with positive integer coefficients. Prove that if n is a positive integer, then $f(n)$ divides $f(f(n)+1)$ if and only if $n = 1$. [Editor's note: one must assume f is nonconstant.]

2 Hints

You don't get hints on a real exam, but these kinds of ideas may help with similar problems. Look these hints if you like, and see if you can make any further progress.

2017 A–2 Work out the first few polynomials, guess a pattern.

2013 A–3 If it is never zero, it cannot change sign.

2010 B–4 Also consider the equation with $x-1$ in place of x.

2008 A–5 Let $P(z) = f(z) + ig(z)$. Show a polynomial with the same degree as P has $n-1$ distinct roots.

2007 B–1 Do arithmetic modulo $f(n)$.

The next page has solutions, don't continue until you want to see them!

3 Solutions

2017 A–2 Define $P_n(x)$ for $P_0(x) = 1$, $P_1(x) = x$, and $P_n(x) = xP_{n-1}(x) - P_{n-2}(x)$. We claim that $P_n(x) = Q_n(x)$ for all $n \geq 0$; since $P_n(x)$ clearly is a polynomial with integer coefficients for all n, this will imply the desired result.

Since $\{P_n\}$ and $\{Q_n\}$ are uniquely determined by their respective recurrence relations and the initial conditions P_0, P_1 or Q_0, Q_1, it suffices to check that $\{P_n\}$ satisfies the same recurrence as Q: that is, $(P_{n-1}(x))^2 - P_n(x)P_{n-2}(x) = 1$ for all $n \geq 2$. Here is one proof of this: for $n \geq 1$, define the 2×2 matrices

$$M_n = \begin{pmatrix} P_{n-1}(x) & P_n(x) \\ P_{n-2}(x) & P_{n-1}(x) \end{pmatrix}, \quad T = \begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix}$$

with $P_{-1}(x) = 0$ (this value being consistent with the recurrence). Then $\det(T) = 1$ and $TM_n = M_{n+1}$, so by induction on n we have

$$(P_{n-1}(x))^2 - P_n(x)P_{n-2}(x) = \det(M_n) = \det(M_1) = 1.$$

2013 A–3 Suppose on the contrary that $a_0 + a_1 y + \cdots + a_n y^n$ is nonzero for $0 < y < 1$. By the intermediate value theorem, this is only possible if $a_0 + a_1 y + \cdots + a_n y^n$ has the same sign for $0 < y < 1$; without loss of generality, we may assume that $a_0 + a_1 y + \cdots + a_n y^n > 0$ for $0 < y < 1$. For the given value of x, we then have

$$a_0 x^m + a_1 x^{2m} + \cdots + a_n x^{(n+1)m} \geq 0$$

for $m = 0, 1, \ldots$, with strict inequality for $m > 0$. Taking the sum over all m is absolutely convergent and hence valid; this yields

$$\frac{a_0}{1-x} + \frac{a_1}{1-x^2} + \cdots + \frac{a_n}{1-x^{n+1}} > 0,$$

a contradiction.

2010 B–4 The pairs (p, q) satisfying the given equation are those of the form $p(x) = ax+b, q(x) = cx+d$ for $a, b, c, d \in \mathbb{R}$ such that $bc - ad = 1$. We will see later that these indeed give solutions.

Suppose p and q satisfy the given equation; note that neither p nor q can be identically zero. By subtracting the equations

$$p(x)q(x+1) - p(x+1)q(x) = 1$$
$$p(x-1)q(x) - p(x)q(x-1) = 1,$$

we obtain the equation

$$p(x)(q(x+1) + q(x-1)) = q(x)(p(x+1) + p(x-1)).$$

The original equation implies that $p(x)$ and $q(x)$ have no common nonconstant factor, so $p(x)$ divides $p(x+1) + p(x-1)$. Since each of $p(x+1)$ and $p(x-1)$ has the same degree and leading coefficient as p, we must have

$$p(x+1) + p(x-1) = 2p(x).$$

If we define the polynomials $r(x) = p(x+1) - p(x)$, $s(x) = q(x+1) - q(x)$, we have $r(x+1) = r(x)$, and similarly $s(x+1) = s(x)$. Put

$$a = r(0), b = p(0), c = s(0), d = q(0).$$

Then $r(x) = a, s(x) = c$ for all $x \in \mathbb{Z}$, and hence identically; consequently, $p(x) = ax + b, q(x) = cx + d$ for all $x \in \mathbb{Z}$, and hence identically. For p and q of this form,

$$p(x)q(x+1) - p(x+1)q(x) = bc - ad,$$

so we get a solution if and only if $bc - ad = 1$, as claimed.

2008 A–5 Form the polynomial $P(z) = f(z) + ig(z)$ with complex coefficients. It suffices to prove that P has degree at least $n-1$.

By replacing $P(z)$ with $aP(z)+b$ for suitable $a, b \in \mathbb{C}$, we can force the regular n-gon to have vertices $\zeta_n, \zeta_n^2, \ldots, \zeta_n^n$ for $\zeta_n = \exp(2\pi i/n)$. It thus suffices to check that there cannot exist a polynomial $P(z)$ of degree at most $n-2$ such that $P(i) = \zeta_n^i$ for $i = 1, \ldots, n$.

We will prove more generally that for any complex number $t \notin \{0,1\}$, and any integer $m \geq 1$, any polynomial $Q(z)$ for which $Q(i) = t^i$ for $i = 1, \ldots, m$ has degree at least $m-1$. If $Q(z)$ has degree d and leading coefficient c, then $R(z) = Q(z+1) - tQ(z)$ has degree d and leading coefficient $(1-t)c$. However, by hypothesis, $R(z)$ has the distinct roots $1, 2, \ldots, m-1$, so we must have $d \geq m-1$.

2007 B–1 The problem fails if f is a constant, e.g., take $f(n) = 1$. We thus assume that f is nonconstant. Write $f(n) = \sum_{i=0}^{d} a_i n^i$ with $a_i > 0$. Then

$$\begin{aligned} f(f(n)+1) &= \sum_{i=0}^{d} a_i (f(n)+1)^i \\ &\equiv f(1) \pmod{f(n)}. \end{aligned}$$

If $n = 1$, then this implies that $f(f(n)+1)$ is divisible by $f(n)$. Otherwise, $0 < f(1) < f(n)$ since f is nonconstant and has positive coefficients, so $f(f(n)+1)$ cannot be divisible by $f(n)$.