

MATH 385 - PROBLEM-SOLVING STRATEGIES - FALL 2025
ASSIGNMENT 3. EXTREMAL PRINCIPLE

1. CLASSICAL EXAMPLES

1. Show that there are no nonzero integer solutions to the equation

$$a^2 + b^2 = 3(x^2 + y^2).$$

2. Let P be a finite family of line segments on a real line such that any two of them share a common point. Prove that all line segments of P share a common point.
3. Prove that every graph G has a bipartite subgraph with at least half as many edges as G .
4. If n points of the plane do not lie on the same line, then there exists a line passing through exactly two points.

2. HOMEWORK ASSIGNMENT

5. Let $n \geq 3$ people sit around a circular table. Suppose that each person's age equals the average of the ages of the two neighbors immediately adjacent to them. Prove that all n people must have the same age.
6. Suppose that in a party with $n \geq 4$ people, we know that for each group of four persons there are either three of them who know each other or three of them such that no two of them know each other. Prove that we can split the people into two rooms, one where every two persons know each other, and another where no two persons know each other.
7. In the plane, n lines are given ($n \geq 3$), no two of them parallel. Through every intersection of two lines there passes at least an additional line. Prove that all lines pass through one point.
8. Consider a finite set of points in the plane with the property that for any three points A, B, C among them, the area of triangle $\triangle ABC$ is at most 1. Prove that all the points lie inside or on the boundary of a triangle of area at most 4.
9. Let P be a family of $mn + 1$ line segments. Prove that P contains either $n + 1$ pairwise intersecting segments or $m + 1$ pairwise disjoint segments.
10. Let S be a closed, bounded set in space such that any plane cut of S is a circle. Prove that S is a sphere.

3. ADDITIONAL PROBLEMS

11. There are n blue points and n red points in the plane, no three of the $2n$ points are collinear. Show that we can pair the blue and red points using n segments that do not cross each other.