Real Function Exercises

Try to solve the problems below. The IMC problems are from Chapter 7 of Daas' book; the remaining problems are from the Putnam. Unless stated otherwise, all variable and function values are real.

1. [7.42, IMC 2006] Let $f : \mathbb{R} \to \mathbb{R}$ be a surjective non-decreasing function. Show f is continuous.

2. [7.36, IMC 2000] Let $f : [0,1] \to [0,1]$ be a strictly increasing function (not necessarily continuous). Show there exists some $x \in [0,1]$ so that $f(x) = x$. What if f is strictly decreasing?

3. [7.30, IMC 1996] Let $f : [0,1] \to [0,1]$ be a continuous function. Consider any sequence given by $x_{n+1} = f(x_n)$. Show that (x_n) converges if and only if $\lim_{n\to\infty}(x_{n+1} - x_n) = 0$.

4. [7.33, IMC 1998] Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function satisfying $f(0) = 2$, $f'(0) = -2$, and $f(1) = 1$. Prove that there exists some $c \in (0,1)$ such that $f(c) \cdot f'(c) + f''(c) = 0$.

5. [7.39, IMC 2004] Let $f, g : [a,b] \to [0,\infty)$ be continuous and non-decreasing functions such that for each $x \in [a,b]$, we have $\int_a^x \sqrt{f(t)}\, dt \leq \int_a^x \sqrt{g(t)}\, dt$, with equality for $x = b$. Prove that $\int_a^b \sqrt{1 + f(t)}\, dt \geq \int_a^b \sqrt{1 + g(t)}\, dt$.

6. [Putnam 2014 B2] Suppose that f is a function on the interval $[1,3]$ such that $-1 \leq f(x) \leq 1$ for all x and $\int_1^3 f(x)\, dx = 0$. How large can $\int_1^3 \frac{f(x)}{x}\, dx$ be?

7. [Putnam 2013 B2] Let $C = \bigcup_{N=1}^{\infty} C_N$, where C_N denotes the set of those 'cosine polynomials' of the form

$$f(x) = 1 + \sum_{n=1}^{N} a_n \cos(2\pi n x)$$

for which: *(i)* $f(x) \geq 0$ for all real x, and *(ii)* $a_n = 0$ whenever n is a multiple of 3.
Determine the maximum value of $f(0)$ as f ranges through C, and prove that this maximum is attained.

8. [Putnam 2012 B1] Let S be a class of functions from $[0,\infty)$ to $[0,\infty)$ that satisfies:

(i) The functions $f_1(x) = e^x - 1$ and $f_2(x) = \ln(x+1)$ are in S;

(ii) If $f(x)$ and $g(x)$ are in S, the functions $f(x) + g(x)$ and $f(g(x))$ are in S;

(iii) If $f(x)$ and $g(x)$ are in S and $f(x) \geq g(x)$ for all $x \geq 0$, then the function $f(x) - g(x)$ is in S.

Prove that if $f(x)$ and $g(x)$ are in S, then the function $f(x)g(x)$ is also in S.

9. [Putnam 2018 A5] Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function satisfying $f(0) = 0$, $f(1) = 1$, and $f(x) \geq 0$ for all $x \in \mathbb{R}$. Show that there exist a positive integer n and a real number x such that $f^{(n)}(x) < 0$.