B-Surprise Putnam Problems

Each problem below is selected from the B session of the Putnam exam, from 2016 to 2019. These were not the easiest problems; rather they were 2nd or 3rd easiest, measured by the number of full-credit solutions received. See if you can make progress on any of these. These problems, solutions, and rankings are all from the Putnam Archive.

Problems:

2015 B–2. Given a list of the positive integers $1,2,3,4,\ldots$, take the first three numbers $1,2,3$ and their sum 6 and cross all four numbers off the list. Repeat with the three smallest remaining numbers $4,5,7$ and their sum 16. Continue in this way, crossing off the three smallest remaining numbers and their sum, and consider the sequence of sums produced: $6,16,27,36,\ldots$. Prove or disprove that there is some number in the sequence whose base 10 representation ends with 2015.

2016 B–2. Define a positive integer n to be *squarish* if either n is itself a perfect square or the distance from n to the nearest perfect square is a perfect square. For example, 2016 is squarish, because the nearest perfect square to 2016 is $45^2 = 2025$ and $2025 - 2016 = 9$ is a perfect square. (Of the positive integers between 1 and 10, only 6 and 7 are not squarish.)

For a positive integer N, let $S(N)$ be the number of squarish integers between 1 and N, inclusive. Find positive constants α and β such that

$$\lim_{N\to\infty} \frac{S(N)}{N^\alpha} = \beta,$$

or show that no such constants exist.

2017 B–3. Suppose that $f(x) = \sum_{i=0}^{\infty} c_i x^i$ is a power series for which each coefficient c_i is 0 or 1. Show that if $f(2/3) = 3/2$, then $f(1/2)$ must be irrational.

2018 B–2. Let n be a positive integer, and let $f_n(z) = n + (n-1)z + (n-2)z^2 + \cdots + z^{n-1}$. Prove that f_n has no roots in the closed unit disk $\{z \in \mathbb{C} \colon |z| \leq 1\}$.

2019 B–5. Let F_m be the mth Fibonacci number, defined by $F_1 = F_2 = 1$ and $F_m = F_{m-1} + F_{m-2}$ for all $m \geq 3$. Let $p(x)$ be the polynomial of degree 1008 such that $p(2n+1) = F_{2n+1}$ for $n = 0,1,2,\ldots,1008$. Find integers j and k such that $p(2019) = F_j - F_k$.

Solutions:

2015 B–2. We will prove that 42015 is such a number in the sequence. Label the sequence of sums $s_0, s_1, \ldots$, and let a_n, b_n, c_n be the summands of s_n in ascending order. We prove the following two statements for each nonnegative integer n:

(a)$_n$ The sequence

$$a_{3n}, b_{3n}, c_{3n}, a_{3n+1}, b_{3n+1}, c_{3n+1}, a_{3n+2}, b_{3n+2}, c_{3n+2}$$

is obtained from the sequence $10n+1, \ldots, 10n+10$ by removing one of $10n+5, 10n+6, 10n+7$.

(b)$_n$ We have

$$\begin{aligned} s_{3n} &= 30n+6, \\ s_{3n+1} &\in \{30n+15, 30n+16, 30n+17\}, \\ s_{3n+2} &= 30n+27. \end{aligned}$$

These statements follow by induction from the following simple observations:

- by computing the table of values

n	a_n	b_n	c_n	s_n
0	1	2	3	6
1	4	5	7	16
2	8	9	10	27

  we see that (a)$_0$ holds;
- (a)$_n$ implies (b)$_n$;
- (a)$_n$ and (b)$_1$, ..., (b)$_n$ together imply (a)$_{n+1}$.

To produce a value of n for which $s_n \equiv 2015 \pmod{10000}$, we take $n = 3m+1$ for some nonnegative integer m for which $s_{3m+1} = 30m+15$. We must also have $30m \equiv 2000 \pmod{10000}$, or equivalently $m \equiv 400 \pmod{1000}$. By taking $m = 1400$, we ensure that $m \equiv 2 \pmod 3$, so $s_m = 10m+7$; this ensures that s_n does indeed equal $30m+15 = 42015$, as desired.

Remark: With a bit more work, we can give a complete description of s_n, and in particular find the first term in the sequence whose decimal expansion ends in 2015.

2016 B–2. We prove that the limit exists for $\alpha = \frac{3}{4}$, $\beta = \frac{4}{3}$.

For any given positive integer n, the integers which are closer to n^2 than to any other perfect square are the ones in the interval $[n^2-n-1, n^2+n]$. The number of squarish numbers in this interval is $1 + \lfloor \sqrt{n-1} \rfloor + \lfloor \sqrt{n} \rfloor$. Roughly speaking, this means that

$$S(N) \sim \int_0^{\sqrt{N}} 2\sqrt{x}\,dx = \frac{4}{3}N^{3/4}.$$

To make this precise, we use the bounds $x-1 \leq \lfloor x \rfloor \leq x$, and the upper and lower Riemann sum estimates for the integral of $\sqrt{x}$, to derive upper and lower bounds on $S(N)$:

$$\begin{aligned} S(N) &\geq \sum_{n=1}^{\lfloor \sqrt{N} \rfloor - 1} (2\sqrt{n-1} - 1) \\ &\geq \int_0^{\lfloor \sqrt{N} \rfloor - 2} 2\sqrt{x}\,dx - \sqrt{N} \\ &\geq \frac{4}{3}(\sqrt{N}-3)^{3/2} - \sqrt{N} \end{aligned}$$

$$
\begin{aligned}
S(N) &\leq \sum_{n=1}^{\lceil \sqrt{N} \rceil} (2\sqrt{n}+1) \\
&\leq \int_0^{\lceil \sqrt{N} \rceil + 1} 2\sqrt{x}\,dx + \sqrt{N} + 1 \\
&\leq \frac{4}{3}(\sqrt{N}+2)^{3/2} + \sqrt{N} + 1.
\end{aligned}
$$

2017 B–3. Suppose by way of contradiction that $f(1/2)$ is rational. Then $\sum_{i=0}^{\infty} c_i 2^{-i}$ is the binary expansion of a rational number, and hence must be eventually periodic; that is, there exist some integers m, n such that $c_i = c_{m+i}$ for all $i \geq n$. We may then write

$$
f(x) = \sum_{i=0}^{n-1} c_i x^i + \frac{x^n}{1-x^m} \sum_{i=0}^{m-1} c_{n+i} x^i.
$$

Evaluating at $x = 2/3$, we may equate $f(2/3) = 3/2$ with

$$
\frac{1}{3^{n-1}} \sum_{i=0}^{n-1} c_i 2^i 3^{n-i-1} + \frac{2^n 3^m}{3^{n+m-1}(3^m - 2^m)} \sum_{i=0}^{m-1} c_{n+i} 2^i 3^{m-1-i};
$$

since all terms on the right-hand side have odd denominator, the same must be true of the sum, a contradiction.

2018 B–2. Note first that $f_n(1) > 0$, so 1 is not a root of f_n. Next, note that

$$
(z-1) f_n(z) = z^n + \cdots + z - n;
$$

however, for $|z| \leq 1$, we have $|z^n + \cdots + z| \leq n$ by the triangle inequality; equality can only occur if $z, \ldots, z^n$ have norm 1 and the same argument, which only happens for $z = 1$. Thus there can be no root of f_n with $|z| \leq 1$.

2019 B–5. We prove that $(j,k) = (2019, 1010)$ is a valid solution. More generally, let $p(x)$ be the polynomial of degree N such that $p(2n+1) = F_{2n+1}$ for $0 \leq n \leq N$. We will show that $p(2N+3) = F_{2N+3} - F_{N+2}$.

Define a sequence of polynomials $p_0(x), \ldots, p_N(x)$ by $p_0(x) = p(x)$ and $p_k(x) = p_{k-1}(x) - p_{k-1}(x+2)$ for $k \geq 1$. Then by induction on k, it is the case that $p_k(2n+1) = F_{2n+1+k}$ for $0 \leq n \leq N-k$, and also that p_k has degree (at most) $N-k$ for $k \geq 1$. Thus $p_N(x) = F_{N+1}$ since $p_N(1) = F_{N+1}$ and p_N is constant.

We now claim that for $0 \leq k \leq N$, $p_{N-k}(2k+3) = \sum_{j=0}^{k} F_{N+1+j}$. We prove this again by induction on k: for the induction step, we have

$$
\begin{aligned}
p_{N-k}(2k+3) &= p_{N-k}(2k+1) + p_{N-k+1}(2k+1) \\
&= F_{N+1+k} + \sum_{j=0}^{k-1} F_{N+1+j}.
\end{aligned}
$$

Thus we have $p(2N+3) = p_0(2N+3) = \sum_{j=0}^{N} F_{N+1+j}$.

Now one final induction shows that $\sum_{j=1}^{m} F_j = F_{m+2} - 1$, and so $p(2N+3) = F_{2N+3} - F_{N+2}$, as claimed. In the case $N = 1008$, we thus have $p(2019) = F_{2019} - F_{1010}$.