

A-Surprise Putnam Problems

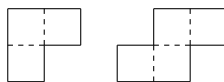
Each problem below is selected from the A session of the Putnam exam, from 2015 to 2020. These were not the easiest problems; rather they were 2nd or 3rd easiest. For 2015–2019, this is based on the number of full-credit solutions received. For 2020 (the unofficial pandemic year), it is my personal opinion.

See if you can make progress on any of these. These problems and solutions are from the [Putnam Archive](#).

Problems:

2015 A–2. Let $a_0 = 1$, $a_1 = 2$, and $a_n = 4a_{n-1} - a_{n-2}$ for $n \geq 2$. Find an odd prime factor of a_{2015} .

2016 A–4. Consider a $(2m-1) \times (2n-1)$ rectangular region, where m and n are integers such that $m, n \geq 4$. This region is to be tiled using tiles of the two types shown:



(The dotted lines divide the tiles into 1×1 squares.) The tiles may be rotated and reflected, as long as their sides are parallel to the sides of the rectangular region. They must all fit within the region, and they must cover it completely without overlapping.

What is the minimum number of tiles required to tile the region?

2017 A–2. Let $Q_0(x) = 1$, $Q_1(x) = x$, and

$$Q_n(x) = \frac{(Q_{n-1}(x))^2 - 1}{Q_{n-2}(x)}$$

for all $n \geq 2$. Show that, whenever n is a positive integer, $Q_n(x)$ is equal to a polynomial with integer coefficients.

2018 A–3. Find the greatest possible value of $\sum_{i=1}^{10} \cos(3x_i)$ for real numbers $x_1, x_2, \dots, x_{10}$ satisfying $\sum_{i=1}^{10} \cos(x_i) = 0$.

2019 A–3. Given real numbers $b_0, b_1, \dots, b_{2019}$ with $b_{2019} \neq 0$, let $z_1, z_2, \dots, z_{2019}$ be the roots in the complex plane of the polynomial

$$P(z) = \sum_{k=0}^{2019} b_k z^k.$$

Let $\mu = (|z_1| + \dots + |z_{2019}|)/2019$ be the average of the distances from $z_1, z_2, \dots, z_{2019}$ to the origin. Determine the largest constant M such that $\mu \geq M$ for all choices of $b_0, b_1, \dots, b_{2019}$ that satisfy

$$1 \leq b_0 < b_1 < b_2 < \dots < b_{2019} \leq 2019.$$

2020 A–3. Let $a_0 = \pi/2$, and let $a_n = \sin(a_{n-1})$ for $n \geq 1$. Determine whether

$$\sum_{n=1}^{\infty} a_n^2$$

converges.