Today we look at some questions from recent Putnam exams, regarding games. However, two player strategy games are not that common, so I have also included some “process games” (predict the outcome of some sequential process). These problems typically involve combinatorics or probability.

As usual: we’ll take about half of our time just to look over the problems individually. Then we’ll discuss any ideas you might have. Then you might look at the hints (second page). We may not have time to go over the solutions, but you can read them later (pages 3+).

I. PROBLEMS

Look over the problems below. Try to identify one or more problems where you have some idea of how to get started. For a real Putnam session, I recommend you spend at least half an hour just on this step!

2017 A5: Each of the integers from 1 to \( n \) is written on a separate card, and then the cards are combined into a deck and shuffled. Three players, \( A \), \( B \), and \( C \), take turns in the order \( A \), \( B \), \( C \), \( A \), . . . choosing one card at random from the deck. (Each card in the deck is equally likely to be chosen.) After a card is chosen, that card and all higher-numbered cards are removed from the deck, and the remaining cards are reshuffled before the next turn. Play continues until one of the three players wins the game by drawing the card numbered 1.

Show that for each of the three players, there are arbitrarily large values of \( n \) for which that player has the highest probability among the three players of winning the game.

2016 B4: Let \( A \) be a \( 2n \times 2n \) matrix, with entries chosen independently at random. Every entry is chosen to be 0 or 1, each with probability \( 1/2 \). Find the expected value of \( \text{det}(A - A^t) \) (as a function of \( n \)), where \( A^t \) is the transpose of \( A \).

2013 B6: Let \( n \geq 1 \) be an odd integer. Alice and Bob play the following game, taking alternating turns, with Alice playing first. The playing area consists of \( n \) spaces, arranged in a line. Initially all spaces are empty. At each turn, a player either

- places a stone in an empty space, or
- removes a stone from a nonempty space \( s \), places a stone in the nearest empty space to the left of \( s \) (if such a space exists), and places a stone in the nearest empty space to the right of \( s \) (if such a space exists).

Furthermore, a move is permitted only if the resulting position has not occurred previously in the game. A player loses if he or she is unable to move. Assuming that both players play optimally throughout the game, what moves may Alice make on her first turn?

2012 B3: A round-robin tournament of \( 2n \) teams lasted for \( 2n - 1 \) days, as follows. On each day, every team played one game against another team, with one team winning and one team losing in each of the \( n \) games. Over the course of the tournament, each team played every other team exactly once. Can one necessarily choose one winning team from each day without choosing any team more than once?

2011 B4: In a tournament, 2011 players meet 2011 times to play a multiplayer game. Every game is played by all 2011 players together and ends with each of the players either winning or losing. The standings are kept in two \( 2011 \times 2011 \) matrices, \( T = (T_{hk}) \) and \( W = (W_{hk}) \). Initially, \( T = W = 0 \). After every game, for every \((h,k)\) (including for \( h = k \)), if players \( h \) and \( k \) tied (that is, both won or both lost), the entry \( T_{hk} \) is increased by 1, while if player \( h \) won and player \( k \) lost, the entry \( W_{hk} \) is increased by 1 and \( W_{kh} \) is decreased by 1.

Prove that at the end of the tournament, \( \text{det}(T + iW) \) is a non-negative integer divisible by \( 2^{2010} \).
II. HINTS

You won’t get hints on a real exam, but these ideas that may help you with similar problems might help. Look these over, and see if you can make any further progress.

**2017 A5:** Derive a recurrence for the probability of player A/B/C winning, if we start with $n$ cards. Then look at the differences between these three probabilities, and how they change (slowly “rotating”) with $n$.

**2016 B4:** A determinant is a sum over permutations, and a permutation is a product of orbits.

**2013 B6:** Say the “weight” of a position is the sum of the labels of the occupied spaces, mod $n+1$. Look for a simple description of the winning positions.

**2012 B3:** Construct a bipartite graph connecting teams and their “winning” days.

**2011 B4:** There is a complex matrix $A$ so that $T + iW = \bar{A}^T A$.

The next page has solutions, don’t continue until you want to see them!