Continuity and Analysis Session

Today we look at some accessible questions from recent Putnam exams, regarding continuity and analysis (convergence, integrals, and so on). There is typically at least one such question each year, although they are not always the easiest. In fact this is probably my weakest area, but maybe some of you will prefer it!

As usual we’ll take about half of our time just to look over the problems individually. Then we’ll discuss any ideas you might have, and how to write them up. After that you might look at the hints (second page). We may not have time to go over the solutions (3rd page).

1. PROBLEMS

Look over the problems below. Try to identify one or more problems where you have some idea of how to get started. For a real Putnam session, I recommend you spend at least half an hour just on this step!

2014 B2: Suppose that \( f \) is a function on the interval \([1, 3]\) such that \( -1 \leq f(x) \leq 1 \) for all \( x \) and \( \int_1^3 f(x) \, dx = 0 \). How large can \( \int_1^3 \frac{f(x)}{x} \, dx \) be?

2013 B2: Let \( C = \bigcup_{N=1}^{\infty} C_N \), where \( C_N \) denotes the set of those ‘cosine polynomials’ of the form

\[
f(x) = 1 + \sum_{n=1}^{N} a_n \cos(2\pi nx)
\]

for which:

(i) \( f(x) \geq 0 \) for all real \( x \), and

(ii) \( a_n = 0 \) whenever \( n \) is a multiple of 3.

Determine the maximum value of \( f(0) \) as \( f \) ranges through \( C \), and prove that this maximum is attained.

2012 B1: Let \( S \) be a class of functions from \([0, \infty)\) to \([0, \infty)\) that satisfies:

(i) The functions \( f_1(x) = e^x - 1 \) and \( f_2(x) = \ln(x + 1) \) are in \( S \);

(ii) If \( f(x) \) and \( g(x) \) are in \( S \), the functions \( f(x) + g(x) \) and \( f(g(x)) \) are in \( S \);

(iii) If \( f(x) \) and \( g(x) \) are in \( S \) and \( f(x) \geq g(x) \) for all \( x \geq 0 \), then the function \( f(x) - g(x) \) is in \( S \).

Prove that if \( f(x) \) and \( g(x) \) are in \( S \), then the function \( f(x)g(x) \) is also in \( S \).

2011 A2: Let \( a_1, a_2, \ldots \) and \( b_1, b_2, \ldots \) be sequences of positive real numbers such that \( a_1 = b_1 = 1 \) and \( b_n = b_{n-1}a_n - 2 \) for \( n = 2, 3, \ldots \). Assume that the sequence \((b_j)\) is bounded. Prove that

\[
S = \sum_{n=1}^{\infty} \frac{1}{a_1 \ldots a_n}
\]

converges, and evaluate \( S \).

2011 B1: Let \( h \) and \( k \) be positive integers. Prove that for every \( \epsilon > 0 \), there are positive integers \( m \) and \( n \) such that

\[
\epsilon < |h\sqrt{m} - k\sqrt{n}| < 2\epsilon.
\]
II. HINTS

You won’t get hints on a real exam, but these ideas that may help you with similar problems might help. Look these over, and see if you can make any further progress.

2014 B2: Compare $f$ to a step function.

2013 B2: Compare $f(0)$ and $f(1/3)$.

2012 B1: Put some other functions in $S$ first.

2011 A2: Hmm, no good hint.

2011 B1: You can put a rational between any two reals.

The next page has solutions, don’t continue until you want to see them!
III. SOLUTIONS

2014 B2: Let \( g(x) \) be 1 for \( 1 \leq x \leq 2 \) and \(-1\) for \( 2 < x \leq 3 \), and define \( h(x) = g(x) - f(x) \). Then \( \int_1^3 h(x) \, dx = 0 \) and \( h(x) \geq 0 \) for \( 1 \leq x \leq 2 \), \( h(x) \leq 0 \) for \( 2 < x \leq 3 \). Now

\[
\int_1^3 \frac{h(x)}{x} \, dx = \int_1^2 \frac{|h(x)|}{x} \, dx - \int_2^3 \frac{|h(x)|}{x} \, dx
\]

and thus \( \int_1^3 \frac{f(x)}{x} \, dx \leq \int_1^3 \frac{g(x)}{x} \, dx = 2 \log 2 - \log 3 = \log \frac{4}{3} \). Since \( g(x) \) achieves the upper bound, the answer is \( \log \frac{4}{3} \).

(Some other solutions are available in the archive.)

2013 B2: We claim that the maximum value of \( f(0) \) is 3. This is attained for \( N = 2, a_1 = \frac{4}{3}, a_2 = \frac{2}{3} \): in this case \( f(x) = 1 + \frac{4}{3} \cos(2\pi x) + \frac{2}{3} \cos(4\pi x) = 1 + \frac{4}{3} \cos(2\pi x) + \frac{2}{3}(2\cos^2(2\pi x) - 1) = \frac{1}{3}(2\cos(2\pi x) + 1)^2 \) is always nonnegative.

Now suppose that \( f = 1 + \sum_{n=1}^N a_n \cos(2\pi nx) \in C \). When \( n \) is an integer, \( \cos(2\pi n/3) \) equals 0 if \( 3|n \) and \(-1/2 \) otherwise. Thus \( a_n \cos(2\pi n/3) = -a_n/2 \) for all \( n \), and \( f(1/3) = 1 - \sum_{n=1}^N (a_n/2) \). Since \( f(1/3) \geq 0, \sum_{n=1}^N a_n \leq 2 \), whence \( f(0) = 1 + \sum_{n=1}^N a_n \leq 3 \).

2012 B1: Each of the following functions belongs to \( S \) for the reasons indicated in the second column.

- \( f(x), g(x) \)
- \( \ln(x+1) \)
- \( \ln(f(x) + 1), \ln(g(x) + 1) \)
- \( \ln(f(x) + 1) + \ln(g(x) + 1) \)
- \( e^x - 1 \)
- \( (f(x) + 1)(g(x) + 1) - 1 \)
- \( f(x)g(x) + f(x) + g(x) \)
- \( f(x) + g(x) \)
- \( f(x)g(x) \)

2011 A2: For \( m \geq 1 \), write

\[
S_m = \frac{3}{2} \left( 1 - \frac{b_1 \cdots b_m}{(b_1 + 2) \cdots (b_m + 2)} \right).
\]

Then \( S_1 = 1 = 1/a_1 \) and a quick calculation yields

\[
S_m - S_{m-1} = \frac{b_1 \cdots b_{m-1}}{(b_2 + 2) \cdots (b_m + 2)} = \frac{1}{a_1 \cdots a_m}
\]

for \( m \geq 2 \), since \( a_j = (b_j + 2)/b_{j-1} \) for \( j \geq 2 \). It follows that \( S_m = \sum_{n=1}^m 1/(a_1 \cdots a_n) \).

Now if \( (b_j) \) is bounded above by \( B \), then \( \frac{b_j}{b_j + 2} \leq \frac{B}{2b_{j+1}} \) for all \( j \), and so \( 3/2 > S_m \geq 3/2(1 - (\frac{B}{2})^m) \). Since \( \frac{B}{2} < 1 \), it follows that the sequence \( S_m \) converges to \( S = 3/2 \).

2011 B1: Since the rational numbers are dense in the reals, we can find positive integers \( a,b \) such that

\[
\frac{3\varepsilon}{hk} < \frac{b}{a} < \frac{4\varepsilon}{hk}.
\]

By multiplying \( a \) and \( b \) by a suitably large positive integer, we can also ensure that \( 3a^2 > b \). We then have

\[
\frac{\varepsilon}{hk} < \frac{b}{3a} < \frac{b}{\sqrt{a^2 + b} + a} = \sqrt{a^2 + b} - a
\]

and

\[
\sqrt{a^2 + b} - a = \frac{b}{\sqrt{a^2 + b} + a} \leq \frac{b}{2a} < \frac{2\varepsilon}{hk}
\]

We may then take \( m = k^2(a^2 + b), n = h^2a^2 \).