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WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

Problems for Session **A**

The 86th William Lowell Putnam Mathematical Competition 2025

A1 Let m_0 and n_0 be distinct positive integers. For every positive integer k, define m_k and n_k to be the relatively prime positive integers such that

$$\frac{m_k}{n_k} = \frac{2m_{k-1}+1}{2n_{k-1}+1}.$$

Prove that $2m_k+1$ and $2n_k+1$ are relatively prime for all but finitely many positive integers k.

A2 Find the largest real number a and the smallest real number b such that

$$ax(\pi - x) \le \sin x \le bx(\pi - x)$$

for all x in the interval $[0, \pi]$.

A3 Alice and Bob play a game with a string of n digits, each of which is restricted to be 0, 1, or 2. Initially all the digits are 0. A legal move is to add or subtract 1 from one digit to create a new string that has not appeared before. A player with no legal move loses, and the other player wins. Alice goes first, and the players alternate moves. For each $n \ge 1$, determine which player has a strategy that guarantees winning.

A4 Find the minimal value of k such that there exist k-by-k real matrices $A_1, \ldots, A_{2025}$ with the property that $A_iA_j = A_jA_i$ if and only if $|i-j| \in \{0, 1, 2024\}$.

A5 Let n be an integer with $n \ge 2$. For a sequence $s = (s_1, \ldots, s_{n-1})$ where each $s_i = \pm 1$, let $f(s)$ be the number of permutations $(a_1, \ldots, a_n)$ of $\{1, 2, \ldots, n\}$ such that $s_i(a_{i+1} - a_i) > 0$ for all i. For each n, determine the sequences s for which $f(s)$ is maximal.

A6 Let $b_0 = 0$ and, for $n \ge 0$, define $b_{n+1} = 2b_n^2 + b_n + 1$. For each $k \ge 1$, show that $b_{2^{k+1}} - 2b_{2^k}$ is divisible by 2^{2k+2} but not by 2^{2k+3}.