The 86th William Lowell Putnam Mathematical Competition 2025

B1 Suppose that each point in the plane is colored either red or green, subject to the following condition: For every three noncollinear points A, B, C of the same color, the center of the circle passing through A, B, and C is also this color. Prove that all points of the plane are the same color.

B2 Let $f\colon [0,1] \to [0,\infty)$ be strictly increasing and continuous. Let R be the region bounded by $x = 0$, $x = 1$, $y = 0$, and $y = f(x)$. Let x_1 be the x-coordinate of the centroid of R. Let x_2 be the x-coordinate of the centroid of the solid generated by rotating R about the x-axis. Prove that $x_1 < x_2$.

B3 Suppose S is a nonempty set of positive integers with the property that if n is in S, then every positive divisor of $2025^n - 15^n$ is in S. Must S contain all positive integers?

B4 For $n \geq 2$, let $A = [a_{i,j}]_{i,j=1}^n$ be an n-by-n matrix of nonnegative integers such that

(a) $a_{i,j} = 0$ when $i + j \leq n$;

(b) $a_{i+1,j} \in \{a_{i,j}, a_{i,j} + 1\}$ when $1 \leq i \leq n-1$ and $1 \leq j \leq n$; and

(c) $a_{i,j+1} \in \{a_{i,j}, a_{i,j} + 1\}$ when $1 \leq i \leq n$ and $1 \leq j \leq n-1$.

Let S be the sum of the entries of A, and let N be the number of nonzero entries of A. Prove that

$$S \leq \frac{(n+2)N}{3}.$$

B5 Let p be a prime number greater than 3. For each $k \in \{1, \ldots, p-1\}$, let $I(k) \in \{1, 2, \ldots, p-1\}$ be such that $k \cdot I(k) \equiv 1 \pmod{p}$. Prove that the number of integers $k \in \{1, \ldots, p-2\}$ such that $I(k+1) < I(k)$ is greater than $p/4 - 1$.

B6 Let $\mathbb{N} = \{1, 2, 3, \ldots\}$. Find the largest real constant r such that there exists a function $g\colon \mathbb{N} \to \mathbb{N}$ such that

$$g(n+1) - g(n) \geq \big(g(g(n))\big)^r$$

for all $n \in \mathbb{N}$.