MATH 385 - PROBLEM-SOLVING STRATEGIES - FALL 2025
ASSIGNMENT 2. DOUBLE COUNTING

Problem 1. In a certain committee, each member belongs to exactly three subcommittees, and each subcommittee has exactly three members. Prove that the number of members equals to the number of subcommittees

Problem 2. Let A be an $m \times n$ matrix with entries in $\{0, 1\}$. Suppose that for every entry $a_{ij} = 1$, the number of 1's in row i equals the number of 1's in column j. Prove that the number of rows containing at least one 1 equals the number of columns containing at least one 1.

Problem 3. Let A be an $m \times n$ matrix with entries in $\{0, 1\}$, where $m < n$. Assume that every column of A contains at least one 1. Prove that there exists an entry $a_{ij} = 1$ such that the number of 1's in row i is strictly larger than the number of 1's in column j.

Problem 4. Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $m \times n$ matrices with entries in $\{0, 1\}$ such that each row of A and B contains exactly one 1. For each row i, let $j(i), k(i) \in \{1, \dots, n\}$ be the unique indices satisfying $a_{i,j(i)} = 1$ and $b_{i,k(i)} = 1$. Suppose that A has exactly k all-zero columns, while every column of B contains at least one 1.

An index $i \in \{1, \dots, m\}$ is called *interesting* if the $k(i)$th column of B contains fewer 1's than the $j(i)$th column of A. Prove that there are at least $k + 1$ interesting indices.

Problem 5. Let $A = (a_{ij})$ be an $n \times n$ matrix. Suppose that for every cell (i, j) with $a_{ij} = 0$, the sum of all entries in its *cross* (that is, all cells in row i and all cells in column j) is at least 1000. Determine the minimal possible sum of all entries in the matrix A provided that: a) Each a_{ij} is either 0 or 1; b) Each a_{ij} is a non-negative integer.

Problem 6 (IMC 2002)**.** Two hundred students participated in a mathematical contest. They had 6 problems to solve. It is known that each problem was correctly solved by at least 120 participants. Prove that there must be two participants such that every problem was solved by at least one of these two students.

Problem 7. Suppose that a convex n-gon and m red points distinct from the vertices of the polygon are drawn on a blackboard. It turns out that each segment between two vertices of the polygon contains at least one red point. Prove the inequality

$$m \geq n \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{\lfloor (n-1)/2 \rfloor}\right).$$

Problem 8. Consider n unit circles drawn in the plane. It is known that each of them intersects with at least one another circle and also there are no two touching circles. It is possible that more than two circles pass through one point. Prove that there are at least n intersection points.

Problem 9. There are n lines in general position in the plane, that is, no three of them share a common point and no two are parallel. These lines split the plane into several parts. Prove that there are at least $n - 2$ triangles among them.