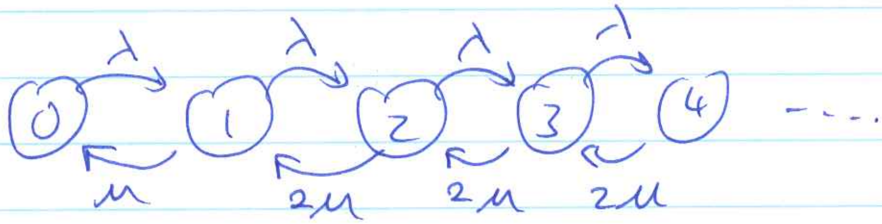
HW2 CS558

Question 1

- State Transition Diagram



- Equi. Eq's:

$$P_1 = \frac{\lambda}{\mu} \cdot P_0 = 2 \cdot \left(\frac{\lambda}{2\mu}\right) P_0.$$

$$P_2 = \frac{\lambda}{2\mu} P_1 = 2 \cdot \left(\frac{\lambda}{2\mu}\right)^2 \cdot P_0$$

$$P_3 = \frac{\lambda}{2\mu} P_2 = 2 \cdot \left(\frac{\lambda}{2\mu}\right)^3 \cdot P_0$$

$$\vdots$$

$$P_K = \frac{\lambda}{2\mu} P_{K-1} = 2 \cdot \left(\frac{\lambda}{2\mu}\right)^K \cdot P_0$$

- Solve steady states using normalization equation:

$$P_0 + P_1 + \ldots + P_K + \ldots = 1$$

$$\Rightarrow P_0 + 2\left(\frac{\lambda}{2\mu}\right) P_0 + 2\left(\frac{\lambda}{2\mu}\right)^2 P_0 + \ldots + 2\left(\frac{\lambda}{2\mu}\right)^K P_0 \ldots = 1.$$

$$\Leftrightarrow P_0 \left(1 + 2\left(\frac{\lambda}{2\mu}\right) + 2\left(\frac{\lambda}{2\mu}\right)^2 + \dots \right) = 1$$

$$\Leftrightarrow P_0 \left(2 - 1 + 2\left(\frac{\lambda}{2\mu}\right) + 2\left(\frac{\lambda}{2\mu}\right)^2 + \dots \right) = 1$$

$$\Leftrightarrow P_0 \left(2 + 2\left(\frac{\lambda}{2\mu}\right) + 2\left(\frac{\lambda}{2\mu}\right)^2 + \dots - 1\right) = 1.$$

$$\Leftrightarrow P_0 \left(2 \underbrace{\left(1 + \left(\frac{\lambda}{2\mu}\right) + \left(\frac{\lambda}{2\mu}\right)^2 + \dots \right)} - 1\right) = 1.$$

$1 + x + x^2 + \dots = \frac{1}{1-x}$

This sum is equal to $\frac{1}{1 - \frac{\lambda}{2\mu}}$.

$$\Leftrightarrow P_0 \left(2 \cdot \frac{1}{1 - \frac{\lambda}{2\mu}} - 1\right) = 1.$$

$$\Leftrightarrow P_0 \left(2 \frac{2\mu}{2\mu - \lambda} - 1\right) = 1.$$

$$\Leftrightarrow P_0 \left(\frac{4\mu}{2\mu - \lambda} - 1\right) = 1$$

$$\Leftrightarrow P_0 \left(\frac{4\mu - (2\mu - \lambda)}{2\mu - \lambda}\right) = 1$$

$$\Leftrightarrow P_0 \left(\frac{2\mu + \lambda}{2\mu - \lambda}\right) = 1$$

$$P_0 = \frac{2\mu - \lambda}{2\mu + \lambda} = \frac{2 - \rho}{2 + \rho} \quad \left(\rho = \frac{\lambda}{\mu}\right)$$

$$= \frac{1 - \frac{1}{2}\rho}{1 + \frac{1}{2}\rho}$$

- Avg. queue length:

$$\overline{Q} = 0.p_0 + 1.p_1 + 2.p_2 + \ldots$$

$$= 0.p_0 + 1 \cdot 2 \cdot \left(\frac{\lambda}{2\mu}\right) p_0 + 2 \cdot 2 \cdot \left(\frac{\lambda}{2\mu}\right)^2 p_0$$

$$+ 3 \cdot 2 \cdot \left(\frac{\lambda}{2\mu}\right)^3 p_0 + \ldots$$

$$= 2 p_0 \left\{ 1 \cdot \left(\frac{\lambda}{2\mu}\right) + 2 \cdot \left(\frac{\lambda}{2\mu}\right)^2 + 3 \left(\frac{\lambda}{2\mu}\right)^3 + \right.$$

$$\left. + \ldots \right\}.$$

From the analysis of the avg. queue length for M/M/1 queue, we found :

$$1 . x + 2 . x^2 + 3 . x^3 + \ldots = x . (1-x)^{-2} \quad (\text{Eq } (3)).$$

$$= \frac{x}{(1-x)^2}.$$

Therefore :

$$\overline{Q} = 2 \cdot p_0 \cdot \frac{\left(\frac{\lambda}{2\mu}\right)}{\left(1 - \frac{\lambda}{2\mu}\right)^2}.$$

$$= 2 \cdot p_0 \cdot \left(\frac{\frac{1}{2}\rho}{(1-\frac{1}{2}\rho)^2} \right).$$

Subst : $P_0 = \frac{1-\frac{1}{2}\rho}{1+\frac{1}{2}\rho}$

We get :

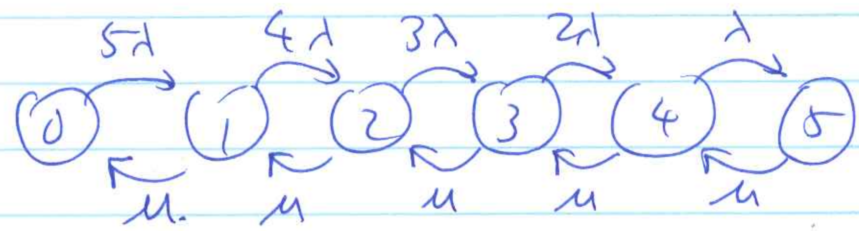
$$\bar{Q} = 2 \cdot \frac{1-\frac{1}{2}\rho}{1+\frac{1}{2}\rho} \cdot \frac{\frac{1}{2}\rho}{(1-\frac{1}{2}\rho)^2} .$$

$$= 2 \cdot \frac{\frac{1}{2}\rho}{(1+\frac{1}{2}\rho)(1-\frac{1}{2}\rho)} .$$

$$= \frac{\rho}{(1+\frac{1}{2}\rho)(1-\frac{1}{2}\rho)} .$$

$$= \frac{\rho}{1-\frac{1}{4}\rho^2} .$$

State transition Diagram:



Equil. Eq's:

$$P_1 = \frac{5\lambda}{\mu} P_0 = 5 \cdot \left(\frac{\lambda}{\mu}\right) P_0 = 5 \cdot \left(\frac{\lambda}{\mu}\right) P_0$$

$$P_2 = \frac{4\lambda}{\mu} P_1 = 5.4 \cdot \left(\frac{\lambda}{\mu}\right)^2 P_0 = 20 \left(\frac{\lambda}{\mu}\right)^2 P_0$$

$$P_3 = 3 \cdot \frac{\lambda}{\mu} P_2 = 5.4.3 \cdot \left(\frac{\lambda}{\mu}\right)^3 \cdot P_0 = 60 \left(\frac{\lambda}{\mu}\right)^3 P_0$$

$$P_4 = 2 \cdot \left(\frac{\lambda}{\mu}\right) P_3 = 5.4.3.2. \left(\frac{\lambda}{\mu}\right)^4 \cdot P_0 = 120 \left(\frac{\lambda}{\mu}\right)^4 P_0$$

$$P_5 = 1 \cdot \left(\frac{\lambda}{\mu}\right) P_4 = 5.4.3.2.1. \left(\frac{\lambda}{\mu}\right)^5 \cdot P_0 = 120 \left(\frac{\lambda}{\mu}\right)^5 P_0$$

Solve: when $\lambda = 1$ and $\mu = 2$

$$5\left(\tfrac{1}{2}\right) P_0 + 20 \left(\tfrac{1}{2}\right)^2 \cdot P_0 + 60 \left(\tfrac{1}{2}\right)^3 P_0 + 120 \left(\tfrac{1}{2}\right)^4 P_0 + 120 \cdot \left(\tfrac{1}{2}\right)^5 \cdot P_0 = 1$$

$$\Rightarrow 2.5 P_0 + 5 \cdot P_0 + 7.5 P_0 + 7.5 P_0 + 3.75 P_0 = 1$$

$$\Leftrightarrow 26.25 P_0 = 1$$

$$\Leftrightarrow P_0 = 0.038095238$$

$P_1 = 5(\frac{1}{2})\, P_0 = 2.5\, P_0 = 0.0952380952$

$P_2 = 20(\frac{1}{2})^2 \cdot P_0 = 5 \cdot P_0 = 0.1904761904$

$P_3 = 60(\frac{1}{2})^3 P_0 = 7.5\, P_0 = 0.2857142857$

$P_4 = 120(\frac{1}{2})^4 \cdot P_0 = 7.5\, P_0 = 0.2857142857$

$P_5 = 120(\frac{1}{2})^5 \cdot P_0 = 3.75\, P_0 = 0.1428571428$

- Avg. queue length when $\lambda = 1$ and $\mu = 2$.

$\overline{Q} = 0 \cdot P_0 + 1 \cdot P_1 + 2 \cdot P_2 + 3 \cdot P_3 + 4 \cdot P_4 + 5 \cdot P_5$

$= 0 \cdot (0.038 \ldots) + 1 \cdot (0.09523 \ldots)$

$+ 2 \cdot (0.19047 \ldots) + 3(0.285714 \ldots)$

$+ 4 \cdot (0.2857 \ldots) + 5 \cdot (0.1428 \ldots)$

$= 3.1904761099$

③

- Avg. arrival rate of break downs (when $\lambda = 1$ and $\mu = 2$)

$$\bar{\lambda} = p_0 \cdot 5 \cdot \lambda + p_1 \cdot 4 \cdot \lambda + p_2 \cdot 3\lambda$$

$$+ p_3 \cdot 2\lambda + p_4 \cdot \lambda + p_5 \cdot 0$$

$$= (0.038\ldots) \cdot 5 \cdot \lambda + (0.0952\ldots) \cdot 4\lambda$$

$$+ (0.1904\ldots) \cdot 3\lambda + (0.2857\ldots) \cdot 2\lambda$$

$$+ (0.2857\ldots) \cdot \lambda + (0.1428\ldots) \cdot 0$$

$$= 2 \cdot \lambda \qquad (\lambda = 1)$$

$$= \underline{\underline{2}}$$ ~~break downs / day.~~

(So if you want to compute the avg. delay, you need to use $\bar{\lambda} = 2$ in Little's formula!)