Putnam Practice Session

Geometry problems (2000–2021)

Here we consider five geometry problems from the Putnam exam (2000 or later). Section I has the problems; Section II has hints; and Section III has brief solutions.

I. Practice Problems

Problem 1: 2000 A-3 (Maximal Octagon Area)

The octagon $P_1P_2P_3P_4P_5P_6P_7P_8$ is inscribed in a circle, with the vertices around the circumference in the given order. Given that the polygon $P_1P_3P_5P_7$ is a square of area 5, and the polygon $P_2P_4P_6P_8$ is a rectangle of area 4, find the maximum possible area of the octagon.

Problem 2: 2003 B-5 (Distance Triangle on a Circle)

Let A, B, and C be equidistant points on the circumference of a circle of unit radius centered at O, and let P be any point in the circle's interior. Let a, b, c be the distances from P to A, B, C, respectively. Show that there is a triangle with side lengths a, b, c, and that the area of this triangle depends only on the distance from P to O.

Problem 3: 2006 A-1 (Volume of a Solid Torus)

Find the volume of the region of points (x, y, z) in $\mathbb{R}^3$ such that
$(\mathrm{x}^2 + y^2 + z^2 + 8)^2 \leq 36(x^2 + y^2)$.

Problem 4: 2019 A-2 (Incenter and Centroid Alignment)

In triangle $\triangle ABC$, let G be the centroid and I be the incenter. Let α and β be the angles at vertices A and B, respectively. Suppose that the segment IG is parallel to AB and that $\beta = 2\arctan(1/3)$. Find α.

Problem 5: 2021 A-3 (Inscribed Regular Tetrahedron)

Determine all positive integers N for which the sphere
$\mathrm{x}^2 + y^2 + z^2 = N$

has an inscribed regular tetrahedron whose vertices have integer coordinates.

II. Hints, Useful Theorems, and Strategies

General Geometric Principles

- **Complex Numbers and Vector Rotations:** Distances to symmetric points on a circle can be mapped into the complex plane using $1, \omega, \omega^2$ (the cube roots of unity). Complex linear combinations like $(x-1)+\omega(x-\omega)+\omega^2(x-\omega^2)=0$ yield vector triangles.

- **Pappus's Centroid Theorem for Solids of Revolution:** The volume of a solid generated by revolving a 2D plane region R about a non-intersecting axis in its plane equals the area of R times the distance traveled by the centroid of R during the revolution ($V = 2\pi\bar{r}A$).

- **Inradius, Altitude, and Centroid Relations:** In any triangle, the distance from a vertex to the opposite side is the altitude h_c. If the segment connecting the incenter I and centroid G is parallel to AB, then the altitude to AB satisfies $h_c = 3r$, where r is the inradius.

- **3D Inscribed Cube & Tetrahedron Duality:** A regular tetrahedron inscribed in a sphere of radius R can be extended to an inscribed cube of side length $s = \sqrt{4N/3}$. The volume of a tetrahedron formed by integer vertex vectors can be calculated via a 3×3 determinant.

Problem-Specific Hints

Hint for Problem 1 (2000 A-3)

Express the area of the octagon as $[P_2P_4P_6P_8] + 2[P_2P_3P_4] + 2[P_4P_5P_6]$. What choice of P_3 and P_5 maximizes the perpendicular distances to the sides of rectangle $P_2P_4P_6P_8$?

Hint for Problem 2 (2003 B-5)

Place A, B, C at $1, \omega, \omega^2$ in the complex plane. Show that $(x-1)$, $\omega(x-\omega)$, and $\omega^2(x-\omega^2)$ form a triangle of side lengths a, b, c, and compute its area using $v_1 \times v_2 = \frac{1}{2}\mathrm{Im}(\overline{v_1}v_2)$.

Hint for Problem 3 (2006 A-1)

Convert to cylindrical coordinates with $r = \sqrt{x^2+y^2}$. Show that the inequality simplifies to $(r-3)^2 + z^2 \leq 1$, and apply Pappus's Centroid Theorem.

Hint for Problem 4 (2019 A-2)

Show that $CD = 3r$, where D is the foot of the altitude from C. Calculate $DB = 4r$ and $EB = 3r$ (where E is the tangency point of the incircle on AB), concluding that $D = A$.

Hint for Problem 5 (2021 A-3)

Show that the four vertices of the tetrahedron, together with their antipodal points, form a cube inscribed in $x^2+y^2+z^2 = N$. Prove that the side length of this cube must be an integer m.

III. Solution Outlines

Solution 1: 2000 A-3

Detailed Derivation: The square $P_1P_3P_5P_7$ has area 5, so its side length is $\sqrt{5}$ and the radius of the circumscribed circle is $R = \sqrt{5/2}$. The rectangle $P_2P_4P_6P_8$ has area 4 and is inscribed in a circle of radius $R = \sqrt{5/2}$. Solving $w \cdot h = 4$ and $w^2 + h^2 = 4R^2 = 10$ gives side lengths $\sqrt{2}$ and $2\sqrt{2}$.
The area of the octagon is given by:
Area $= [\mathrm{P}_2P_4P_6P_8$
$+ 2[\mathrm{P}_2P_3P_4] + 2[P_4P_5P_6]$.
$]The area[\mathrm{P}_2P_3P_4]$ is maximized when P_3 is the midpoint of arc P_2P_4, giving height $R - \frac{\sqrt{2}}{2} = \frac{\sqrt{5}-\sqrt{2}}{2}$. Similarly, $[P_4P_5P_6]$ is maximized when P_5 is the midpoint of arc P_4P_6, giving height $R - 1 = \frac{\sqrt{5}-2}{2}$.
Calculating the maximum total area:
$\text{Area}_{\max} = 4 + 2\left(\frac{1}{2} \cdot 2\sqrt{2} \cdot \frac{\sqrt{5}-\sqrt{2}}{2}\right) + 2\left(\frac{1}{2} \cdot \sqrt{2} \cdot \frac{\sqrt{5}-2}{2}\right) = 3\sqrt{5}.$

Solution 2: 2003 B-5

Detailed Derivation: Place A, B, C at $1, \omega, \omega^2$ where $\omega = e^{2\pi i/3}$. Let P correspond to $x \in \mathbb{C}$ with $|x| < 1$. The vector identity $(x-1) + \omega(x-\omega) + \omega^2(x-\omega^2) = 0$ proves that the three segments of lengths $a = |x-1|$, $b = |x-\omega|$, $c = |x-\omega^2|$ form a closed triangle.
To compute the area of this triangle, take two vertex vectors $v_1 = x - 1$ and $v_2 = \omega(x-\omega)$:
Area $= 1\frac{}{4|\overline{v_1}v_2 - v_1\overline{v_2}| = \frac{\sqrt{3}}{4}(1-|x|^2).}$

Since $|x|$ is the distance PO, the area depends solely on the distance from P to the center O.

Solution 3: 2006 A-1

Detailed Derivation: In cylindrical coordinates (r, θ, z) with $r = \sqrt{x^2 + y^2}$, the inequality becomes:
$(\mathrm{r}^2 + z^2 + 8)^2 \le 36r^2 \implies r^2 + z^2 + 8 \le 6r \implies (r-3)^2 + z^2 \le 1.$

This defines a cross-sectional disk of radius 1 centered at $(r, z) = (3, 0)$ in the rz-plane. Revolving this disk about the z-axis creates a solid torus. By Pappus's Centroid Theorem:
V = (Area of disk) $\times$(Distance traveled by centroid) $= (\pi \cdot 1^2) \times (2\pi \cdot 3) = 6\pi^2$.

Solution 4: 2019 A-2

Detailed Derivation: Since $IG \parallel AB$, the distance from C to AB (altitude $h_c = CD$) is 3 times the distance from G to AB (r), so $CD = 3r$. Using $\beta = 2\arctan(1/3)$, we have $\tan(\beta/2) = 1/3$ and $\tan\beta = \frac{2(1/3)}{1-1/9} = 3/4$. In right triangle $\triangle CDB$, $DB = CD/\tan\beta = 3r/(3/4) = 4r$. Let E be the point of tangency of the incircle on AB. Then $EB = r/\tan(\beta/2) = 3r$. Since $DB - EB = 4r - 3r = r$, D lies at distance r from E, placing D directly below the center of the incircle. Thus, CD is tangent to the incircle, forcing the altitude foot D to coincide with vertex A. Therefore, $\triangle ABC$ is a right triangle at A, so $\alpha = \pi/2$.

Solution 5: 2021 A-3

Detailed Derivation: Let P_1, P_2, P_3, P_4 be the integer vertices of an inscribed regular tetrahedron. The centroid of this regular tetrahedron is at the origin $(0,0,0)$. Combining these four vertices with their antipodes $Q_i = -P_i$ forms the eight vertices of a cube inscribed in the sphere $x^2 + y^2 + z^2 = N$.
The volume of this cube is $s^3 = (4N/3)^{3/2}$. On the other hand, the volume of the parallelepiped/cube formed by three edge vectors $Q_2 - Q_1, Q_3 - Q_1, Q_4 - Q_1$ is given by the determinant of a 3×3 integer matrix, which must be an integer $V \in \mathbb{Z}^+$. Thus, $(4N/3)^3 = V^2$ is a perfect square, which implies $N/3 = m^2$ for some integer m. Hence, N must be of the form $N = 3m^2$ for some positive integer m.