Putnam Practice Session

Number Theory problems

Here we consider five number theory problems from the Putnam exam (2000 or later). Section I has the problems, Section II has hints, and Section III has solutions.

I. Practice Problems

Problem 1: 2001 A-5 (Exponential Diophantine Equation)

Prove that there are unique positive integers a, n such that $a^{n+1} - (a+1)^n = 2001$.

Problem 2: 2007 B-1 (Polynomial Divisibility)

Let f be a nonconstant polynomial with positive integer coefficients. Prove that if n is a positive integer, then $f(n)$ divides $f(f(n)+1)$ if and only if $n = 1$.

Problem 3: 2017 B-2 (Consecutive Integer Sums)

Suppose that a positive integer N can be expressed as the sum of k consecutive positive integers
N = a + (a+1) + (a+2) + ... + (a+k-1)

for $k = 2017$ but for no other values of $k > 1$. Considering all positive integers N with this property, what is the smallest positive integer a that occurs in any of these expressions?

Problem 4: 2021 A-5 (Coprime Power Sums Modulo 2021)

Let A be the set of all integers n such that $1 \leq n \leq 2021$ and $\gcd(n, 2021) = 1$. For every nonnegative integer j, let
S(j) = $\sum_{n \in A} n^j$.

Determine all values of j such that $S(j)$ is a multiple of 2021.

Problem 5: 2024 A-1 (Diophantine Power Equation)

Determine all positive integers n for which there exist positive integers a, b, and c satisfying $a^n + 3b^n = 4c^n$.

II. Hints, Useful Theorems, and Strategies

General Number-Theoretic Principles

- **Modular Reduction and Parity Constraints:** Reducing Diophantine equations modulo small integers (e.g., $3, 4, 8$) often restricts variable parities or forces exponents to take specific modular forms.
- **Polynomial Congruences:** For any polynomial $f(x)$ with integer coefficients, $a \equiv b \pmod{m}$ implies $f(a) \equiv f(b) \pmod{m}$. In particular, $f(x + k \cdot f(x)) \equiv f(x) \pmod{f(x)}$.
- **Factorization of Consecutive Sums:** The sum of k consecutive integers starting at a satisfies $2N =$. Factoring $2N$ into two odd/even factors of opposite parity characterizes all possible run lengths k.
- **Power Sums over Finite Fields:** For a prime p, the sum $\sum_{n=1}^{p-1} n^j \pmod{p}$ is congruent to -1 if $(p-1) \mid j$, and congruent to 0 if $(p-1) \nmid j$.
- **Infinite Descent and Valuations:** When quadratic residues or p-adic valuations force all terms in a Diophantine equation to be divisible by p, repeated division demonstrates that no minimal positive integer solution exists.

Problem-Specific Hints

Hint for Problem 1 (2001 A-5)

Analyze the equation $a^{n+1} - (a+1)^n = 2001$ modulo 3 to show that $a \equiv 1 \pmod{3}$ and n must be even. Then analyze modulo 4 to deduce $a \equiv 1 \pmod{4}$.

Hint for Problem 2 (2007 B-1)

Write $f(x) = \sum_{i=0}^{d} a_i x^i$ with $a_i > 0$. Use polynomial congruences modulo $f(n)$ to evaluate $f(f(n)+1) \pmod{f(n)}$ and compare the size of $f(1)$ with $f(n)$.

Hint for Problem 3 (2017 B-2)

Express $2N = k(2a + k - 1)$. Since $k = 2017$ is prime, what condition must $2a + 2016$ satisfy so that no other non-trivial factorizations of $2N$ yield valid length $k > 1$?

Hint for Problem 4 (2021 A-5)

Note $2021 = 43 \times 47$. Use Chinese Remainder Theorem to split $S(j) \pmod{2021}$ into conditions modulo 43 and 47.

Hint for Problem 5 (2024 A-1)

For $n = 1$, test small values. For $n = 2$, examine quadratic residues modulo 3. For $n \geq 3$, examine 2-adic valuations of each term in $2a^n + 3b^n = 4c^n$.

III. Solution Outlines

Solution 1: 2001 A-5

Detailed Derivation: Suppose $a^{n+1}-(a+1)^n = 2001$. Since 2001 is divisible by 3, reducing modulo 3 gives $a^{n+1}-(a+1)^n \equiv 0 \pmod 3$. If $a \equiv 0 \pmod 3$, then $0-1 \equiv -1 \not\equiv 0$. If $a \equiv 2 \pmod 3$, then $a+1 \equiv 0$, so $2^{n+1} \equiv 0$, impossible. Thus $a \equiv 1 \pmod 3$. Then $1-2^n \equiv 0 \pmod 3 \implies 2^n \equiv 1 \pmod 3$, which forces n to be even.
Since n is even, $n \geq 2$. Reducing modulo 4: if a were even, $a^{n+1}-(a+1)^n \equiv 0-1 \equiv 3 \pmod 4$, but $2001 \equiv 1 \pmod 4$. Thus a must be odd. Since a is odd and $a \equiv 1 \pmod 3$, a must divide $2002 = 2 \times 7 \times 11 \times 13$. Furthermore, $a \equiv 1 \pmod 4$ and $11 \nmid a$ (since divisors congruent to 1 (mod 3) cannot be divisible by 11). Testing divisors of 7×13 congruent to 1 (mod 4) yields $a = 13$.
Plugging in $a = 13$: $13^{n+1} - 14^n = 2001$. Testing $n = 2$ gives $13^3 - 14^2 = 2197 - 196 = 2001$, which works. For $n > 2$ even, $13^{n+1} \equiv 2001 \equiv 1 \pmod 8$, but $13^{n+1} \equiv 5^{n+1} \equiv 5 \pmod 8$, a contradiction. Thus, the unique solution is $a = 13, n = 2$.

Solution 2: 2007 B-1

Detailed Derivation: Let $f(x) = \sum_{i=0}^{d} a_i x^i$ with $a_i > 0$. Reducing modulo $f(n)$:
f(f(n) + 1) $= \sum_{i=0}^{d} a_i (f(n)+1)^i \equiv \sum_{i=0}^{d} a_i (1)^i = f(1) \pmod{f(n)}$.

Thus $f(n) \mid f(f(n)+1) \iff f(n) \mid f(1)$. Since all coefficients a_i are positive integers and f is nonconstant:

- If $n = 1$, then $f(1) \mid f(1)$, which is trivially true.
- If $n > 1$, then $0 < f(1) < f(n)$. A smaller positive integer cannot be divisible by a strictly larger positive integer, so $f(n) \nmid f(1)$.

Hence, $f(n)$ divides $f(f(n)+1)$ if and only if $n = 1$.

Solution 3: 2017 B-2

Detailed Derivation: The sum of k consecutive integers starting at a is $N = \frac{k(2a+k-1)}{2}$, so $2N = k(2a + k - 1)$. Here $2a + k - 1 > k > 1$, and k and $2a + k - 1$ have opposite parity (one is odd, the other even). Every valid representation of N as a sum of $k > 1$ consecutive integers corresponds to choosing an odd factor $m > 1$ of $2N$:

- If $m < \sqrt{2N}$, set $k = m$ and $2a + k - 1 = 2N/m$.
- If $m > \sqrt{2N}$, set $k = 2N/m$ and $2a + k - 1 = m$.

We are given that $k = 2017$ is the *only* allowed length $k > 1$. Since 2017 is prime, $2N$ can have no odd prime factors other than 2017, and 2017 must appear with exponent 1. Thus $2N = 2017 \times 2^m$ for some integer $m \geq 1$. To prevent any other odd factor from creating another valid length $k > 1$, $2a + 2016$ must have no odd divisors greater than 1. Therefore, $2a + 2016$ must be a power of 2:
a + 2016 = 2^m.

The smallest power of 2 strictly greater than 2016 is $2^{11} = 2048$. Solving $2a + 2016 = 2048$ yields $2a = 32 \implies a = 16$.

Solution 4: 2021 A-5

Detailed Derivation: Note that $2021 = 43 \times 47$, where 43 and 47 are primes. By the Chinese Remainder Theorem, $S(j) \equiv 0 \pmod{2021}$ if and only if $S(j) \equiv 0 \pmod{43}$ and $S(j) \equiv 0 \pmod{47}$.
Modulo 43, the elements in A distributed modulo 43 cover each non-zero residue in $\{1, \ldots, 42\}$ exactly 46 times. Since $\gcd(46, 43) = 1$:
S(j) $\equiv 46 \sum_{n=1}^{42} n^j \equiv 0 \pmod{43} \iff \sum_{n=1}^{42} n^j \equiv 0 \pmod{43} \iff j \not\equiv 0 \pmod{42}$.

Modulo 47, the elements in A distributed modulo 47 cover each non-zero residue in $\{1, \ldots, 46\}$ exactly 42 times. Since $\gcd(42, 47) = 1$:
S(j) $\equiv 42 \sum_{n=1}^{46} n^j \equiv 0 \pmod{47} \iff \sum_{n=1}^{46} n^j \equiv 0 \pmod{47} \iff j \not\equiv 0 \pmod{46}$.

Thus, $S(j)$ is a multiple of 2021 if and only if j is not divisible by 42 and j is not divisible by 46.

Solution 5: 2024 A-1

Detailed Derivation: For $n = 1$, setting $(a, b, c) = (1, 2, 2)$ gives $2(1) + 3(2) = 8 = 4(2)$, which is valid.

For $n = 2$, suppose $2a^2 + 3b^2 = 4c^2$ with $\gcd(a, b, c) = 1$. Reducing modulo 3 gives $2a^2 \equiv c^2 \pmod 3 \implies a^2 + c^2 \equiv 0 \pmod 3$. Since quadratic residues modulo 3 are 0 and 1, we must have $a \equiv c \equiv 0 \pmod 3$. Then $3 \mid a$ and $3 \mid c$, so $3b^2 = 4c^2 - 2a^2$ is divisible by 9, forcing $3 \mid b$. This contradicts $\gcd(a, b, c) = 1$.

For $n \geq 3$, suppose $2a^n + 3b^n = 4c^n$ with $\gcd(a, b, c) = 1$. Since $3b^n = 4c^n - 2a^n$, b^n must be even, so $2 \mid b$. Then $a^n + 2^{n-1} \cdot 3(b/2)^n = 2c^n$, so a^n must be even, implying $2 \mid a$. Then $2^{n-1}(a/2)^n + 2^{n-2} \cdot 3(b/2)^n = c^n$. Since $n \geq 3$, $n - 2 \geq 1$, so the left side is even, forcing $2 \mid c$. This contradicts $\gcd(a, b, c) = 1$.

Thus, $n = 1$ is the only positive integer for which solutions exist.