More Number Theory Exercises

Sasha contributed two more number theory problems for today (9/18/2026).

1. [IMC 2014, Day 1 P4]
Call a positive integer m *perfect* if the sum of its positive divisors equals $2m$. Suppose $n > 6$ is perfect, and write its prime factorization as

$$n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}, \qquad p_1 < p_2 < \cdots < p_k,$$

where every e_i is positive. Show that the exponent e_1 of the smallest prime divisor is even.

2. [Putnam 2013, A2]
For each positive integer n that is not a square, consider all finite increasing lists of integers

$$n < a_1 < a_2 < \cdots < a_r, \qquad r \geq 1,$$

for which $n a_1 a_2 \cdots a_r$ is a perfect square. Define $f(n)$ to be the least attainable value of the last entry a_r. Prove that f is injective on the set of positive integers that are not squares.