Putnam Practice

Invariants, Parity, and Coloring Arguments

Overview & Pedagogical Goals

We focus on three related problem-solving tools:

1. **Invariants and Monovariants**: Quantities or structural properties that remain unchanged (or change monotonically) under specified operations or state transitions.
2. **Parity and Pairing (Involutions)**: Exploiting even/odd parity constraints or constructing fixed-point-free involutions to demonstrate existence or parity of solution counts.
3. **Coloring Arguments**: Partitioning geometric structures or discrete grids into colored subsets to establish impossible configurations or tight combinatorial bounds.

1 Study Guide & Core Principles

1.1 1. Invariants and Monovariants

When analyzing discrete processes, multi-step games, or operational transformations, tracking the full state space is often intractable. Instead, we search for a scalar or structural property $I(S)$ evaluated on state S such that $I(S_{\text{initial}}) = I(S_{\text{next}})$, or a quantity $M(S)$ that changes strictly monotonically to guarantee termination.

Definition 1 (Invariant and Monovariant)**.** *An **invariant** is a function $I : \Omega \to \mathcal{X}$ from state space Ω to a set $\mathcal{X}$ such that for any valid state transition $S \to S'$, $I(S) = I(S')$. If $I(S_{initial}) \neq I(S_{target})$, then S_{target} is unreachable from $S_{initial}$.*
*A **monovariant** is a function $M : \Omega \to \mathbb{R}$ such that $M(S') > M(S)$ (or $M(S') < M(S)$) for every step. If M is bounded, the process must terminate in finitely many steps.*

Example 1 (The Swap / Parity Invariant)**.** *Suppose n numbers are placed in a row. An operation consists of picking two adjacent numbers and swapping them if they are out of order. The total number of inverted pairs modulo 2 changes parity with every single swap, while the total sum of the numbers is a strict algebraic invariant.*

Example 2 (Algebraic Polynomial Invariant)**.** *Suppose the numbers $x_1, x_2, \ldots, x_n$ are written on a blackboard. In each step, we erase any two numbers a and b and replace them with $a + b + ab$.*
Notice that $a + b + ab = (a + 1)(b + 1) - 1$. Therefore, the product $P = \prod_{i=1}^{n}(x_i + 1)$ is an exact invariant of the process. Regardless of the order of operations, the final single remaining number X will always satisfy $X + 1 = \prod_{i=1}^{n}(x_i + 1)$.

Example 3 (Monovariants and Matrix Sign-Flipping). *Consider an $n \times n$ matrix of real numbers. An allowed move consists of picking any row or column whose sum is negative and multiplying all entries in that row or column by -1.*
To prove this process must terminate, define the monovariant $M(S) = \sum_{i,j} a_{i,j}$ as the sum of all entries in the matrix. Flipping a row or column with negative sum $R < 0$ changes its sum to $-R > 0$, increasing $M(S)$ by $2|R| > 0$. Since there are only 2^{n^2} possible sign assignments for the matrix entries, $M(S)$ is bounded above, so the process terminates in finitely many steps.

1.2 2. Parity and Involutions (Pairing Arguments)

Many counting problems ask whether a set size $|S|$ is even or odd, or require showing that $|S| \equiv |T| \pmod 2$. A primary method is constructing a **pairing** (an involution) without fixed points, or counting fixed points modulo 2.

Theorem 1 (Involution Principle). *Let S be a finite set and $f : S \to S$ be a bijection such that $f(f(x)) = x$ for all $x \in S$ (an involution). Let $F = \{x \in S : f(x) = x\}$ be the set of fixed points. Then:*

$$|S| \equiv |F| \pmod 2.$$

In particular, if f has no fixed points ($F = \emptyset$), then $|S|$ is an even integer.

Example 4 (Divisor Parity). *The number of positive divisors $d(n)$ of an integer n is odd if and only if n is a perfect square. This follows from the involution $f(d) = n/d$ on the set of divisors of n. The fixed points satisfy $d = n/d \iff d^2 = n$, which exists if and only if n is a square.*

Example 5 (The Handshaking Lemma). *In any finite graph $G = (V, E)$, the number of vertices with odd degree is always even.*
Consider the set of incident vertex-edge pairs $S = \{(v, e) : v \in V, e \in E, v \text{ is an endpoint of } e\}$. Summing over vertices gives $|S| = \sum_{v \in V} \deg(v)$. Summing over edges, each edge has exactly 2 endpoints, so $|S| = 2|E|$, which is even. Parity requires that $\sum_{v \in V} \deg(v) \equiv 0 \pmod 2$, forcing an even number of odd-degree vertices.

Example 6 (Complementary Subset Pairing). *For any finite set $A = \{1, 2, \ldots, n\}$ with $n \geq 1$, the map $f(S) = A \setminus S$ is a fixed-point-free involution on the power set $\mathcal{P}(A)$ (since $S = A \setminus S \implies A = \emptyset$, a contradiction for $n \geq 1$). Thus, the total number of subsets 2^n is even, and subsets naturally pair into disjoint complement pairs (S, S^c).*

1.3 3. Coloring Arguments

Coloring arguments assign labels or colors to elements of a grid, graph, or discrete structure to exploit local tiling or coverage constraints.

Definition 2 (Coloring Bound). *If a region contains C_{red} red cells, and every tile of type T can cover at most k red cells under allowed placements, then any valid tiling of the region using tiles of type T requires at least $\lceil C_{red}/k \rceil$ tiles.*

Example 7 (The Chessboard Defect). *An 8×8 chessboard with two diagonally opposite corners removed cannot be tiled by 1×2 dominoes. Removing two opposite corners removes two squares of the same color (say, two white squares), leaving 32 black and 30 white squares. Since each 1×2 domino must cover exactly 1 black and 1 white square, any set of dominoes covers an equal number of black and white squares, making a complete tiling impossible.*

Example 8 (Multi-Color Stripe Tiling). *Can a 10×10 board be tiled using twenty-five 1×4 straight tetrominoes?*
Color the columns of the 10×10 board with 4 colors in repeating order: $1, 2, 3, 4, 1, 2, 3, 4, 1, 2$.

- *Columns colored 1 and 2 appear 3 times (30 cells each).*
- *Columns colored 3 and 4 appear 2 times (20 cells each).*

Any horizontal 1×4 tetromino covers exactly one cell of each of the 4 colors. Any vertical 1×4 tetromino lies entirely within a single column, covering 4 cells of the same color.
Thus, every tile covers a multiple of 4 cells of color 3 plus an equal number of cells of all other colors horizontally. Modulo 2, each tile covers an even number of cells of each color. However, color 1 has 30 cells (even) and color 3 has 20 cells (even), but checking modulo 4 shows that $30 \not\equiv 20 \pmod 4$. Since horizontal tiles cover equal numbers of each color and vertical tiles cover 4 of a single color, a full tiling is impossible.

2 Selected Putnam Problems

Here are five Putnam problems whose solutions rely on such ideas.

Problem 1 (Putnam 2001, Problem B1)

Let n be an even positive integer. Write the numbers $1, 2, \ldots, n^2$ in the squares of an $n \times n$ grid so that the k-th row, from left to right, is

$$(k-1)n + 1,\ (k-1)n + 2,\ \ldots,\ (k-1)n + n.$$

Color the squares of the grid so that half of the squares in each row and in each column are red and the other half are black. Prove that for each coloring, the sum of the numbers on the red squares is equal to the sum of the numbers on the black squares.

Problem 2 (Putnam 2002, Problem A3)

Let $n \geq 2$ be an integer and T_n be the number of nonempty subsets S of $\{1, 2, \ldots, n\}$ with the property that the average of the elements of S is an integer. Prove that $T_n - n$ is always even.

Problem 3 (Putnam 2003, Problem A6)

For a set S of nonnegative integers, let $r_S(n)$ denote the number of ordered pairs (s_1, s_2) such that $s_1 \in S, s_2 \in S, s_1 \neq s_2$, and $s_1 + s_2 = n$. Is it possible to partition the nonnegative integers into two sets A and B in such a way that $r_A(n) = r_B(n)$ for all n?

Problem 4 (Putnam 2016, Problem A4)

Consider a $(2m-1) \times (2n-1)$ rectangular region, where $m, n \geq 4$. This region is to be tiled using tiles of two types: L-shaped 4-unit tiles and 4-unit square/tetromino tiles. The tiles may be rotated and reflected, as long as their sides are parallel to the sides of the rectangular region. They must all fit within the region and cover it completely without overlapping. What is the minimum number of tiles required to tile the region?

Problem 5 (Putnam 2018, Problem B1)

Let P be the set of vectors defined by

$$P = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} : 0 \leq a \leq 2,\ 0 \leq b \leq 100,\ \text{and } a, b \in \mathbb{Z} \right\}.$$

Find all $v \in P$ such that the set $P \setminus \{v\}$ obtained by omitting vector v from P can be partitioned into two sets of equal size and equal sum.

3 Solutions and Commentary

Solution to Problem 1 (Putnam 2001, B1)

Proof. Let R denote the set of red squares and B denote the set of black squares in the grid. For any square $s \in R \cup B$ located at row i and column j (where $0 \le i, j \le n-1$), the number written in square s is given by

$$v(s) = i \cdot n + j + 1.$$

Let $f(s) = i$ denote the row index of square s, and let $g(s) = j$ denote the column index of square s. Then $v(s) = nf(s) + g(s) + 1$.

Since every row contains exactly $n/2$ red squares and $n/2$ black squares, the sum of row indices over all red squares equals the sum of row indices over all black squares:

$$\sum_{s \in R} f(s) = \sum_{s \in B} f(s) = \frac{n}{2} \sum_{i=0}^{n-1} i.$$

Similarly, because every column contains exactly $n/2$ red squares and $n/2$ black squares, the sum of column indices over red squares equals that over black squares:

$$\sum_{s \in R} g(s) = \sum_{s \in B} g(s) = \frac{n}{2} \sum_{j=0}^{n-1} j.$$

Finally, noting that $|R| = |B| = n^2/2$, we compute the total sum of numbers on the red squares:

$$\sum_{s \in R} v(s) = n \sum_{s \in R} f(s) + \sum_{s \in R} g(s) + |R| \cdot 1 = n \sum_{s \in B} f(s) + \sum_{s \in B} g(s) + |B| \cdot 1 = \sum_{s \in B} v(s).$$

Thus, the sum of the numbers on the red squares is identically equal to the sum on the black squares for every valid coloring. □

Solution to Problem 2 (Putnam 2002, A3)

Proof. Let $\mathcal{A}$ be the collection of all nonempty subsets $S \subseteq \{1, 2, \ldots, n\}$ whose elements have an integer average $m(S) \in \mathbb{Z}$. By definition, $T_n = |\mathcal{A}|$.

Notice that each of the n singleton sets $\{1\}, \{2\}, \ldots, \{n\}$ belongs to $\mathcal{A}$, because the average of $\{k\}$ is simply $k \in \mathbb{Z}$.

Now consider the remaining elements of $\mathcal{A}$, namely subsets $S \in \mathcal{A}$ with $|S| \ge 1$ that are not singletons unless $|S| = 1$. Define a mapping $f : \mathcal{A} \setminus \{\{1\}, \ldots, \{n\}\} \to \mathcal{A} \setminus \{\{1\}, \ldots, \{n\}\}$ as follows: Let $m = m(S)$ be the integer average of S.

- If $m \notin S$, set $f(S) = S \cup \{m\}$.
- If $m \in S$ (and $|S| > 1$), set $f(S) = S \setminus \{m\}$.

We check that $f(S)$ preserves the average m: If S has size k and sum km, then $S \cup \{m\}$ has size $k+1$ and sum $km + m = (k+1)m$, so its average is still m. Conversely, $S \setminus \{m\}$ has size $k-1$ and sum $km - m = (k-1)m$, so its average is also m.

Furthermore, $f(f(S)) = S$, so f is a fixed-point-free involution on $\mathcal{A} \setminus \{\{1\}, \ldots, \{n\}\}$. Therefore, all subsets in $\mathcal{A}$ other than the n singletons can be grouped into disjoint pairs $\{S, f(S)\}$.

It follows that $T_n - n = |\mathcal{A} \setminus \{\{1\}, \ldots, \{n\}\}|$ is an even integer. □

Solution to Problem 3 (Putnam 2003, A6)

Proof. Yes, such a partition is possible.

Define $b(k)$ to be the number of 1s in the binary representation of k. Partition the nonnegative integers $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ into two sets A and B:

$$A = \{k \in \mathbb{N}_0 : b(k) \text{ is even}\}, \qquad B = \{k \in \mathbb{N}_0 : b(k) \text{ is odd}\}.$$

To show that $r_A(n) = r_B(n)$ for all $n \geq 0$, we exhibit a bijection between pairs of distinct elements $(a_1, a_2) \in A \times A$ with $a_1 + a_2 = n$ and pairs $(b_1, b_2) \in B \times B$ with $b_1 + b_2 = n$.

Given a pair (a_1, a_2) of distinct elements in A with $a_1 + a_2 = n$, express both a_1 and a_2 in binary. Since $a_1 \neq a_2$, there is a lowest-order bit position 2^j where a_1 and a_2 differ. One of the numbers has a 0 at 2^j and the other has a 1 at 2^j.

Invert the bit at position 2^j for both numbers, yielding b_1 and b_2. Because one number had a 0 changed to 1 and the other had a 1 changed to 0, the sum is preserved:

$$b_1 + b_2 = (a_1 \pm 2^j) + (a_2 \mp 2^j) = a_1 + a_2 = n.$$

However, flipping a single bit changes the parity of the number of 1s in a_1 and a_2. Since $b(a_1)$ and $b(a_2)$ were both even, $b(b_1)$ and $b(b_2)$ are both odd, so $b_1, b_2 \in B$.

This map is its own inverse and establishes a bijection between valid pairs in $A \times A$ and valid pairs in $B \times B$. Hence $r_A(n) = r_B(n)$ for all n. □

Solution to Problem 4 (Putnam 2016, A4)

Proof. The minimum number of tiles required is mn.

Lower Bound: Label the unit squares of the $(2m-1) \times (2n-1)$ grid as (i, j) for $1 \leq i \leq 2m-1$ and $1 \leq j \leq 2n-1$. Color square (i, j) **red** if and only if both i and j are odd.

The number of odd integers in $\{1, 2, \dots, 2m-1\}$ is m, and in $\{1, 2, \dots, 2n-1\}$ is n. Thus, there are exactly $m \times n = mn$ red squares..

Observe that any two red squares (i_1, j_1) and (i_2, j_2) satisfy $|i_1 - i_2| \geq 2$ or $|j_1 - j_2| \geq 2$. Thus, no single tile of 4 units (of either type) can cover more than one red square. Since every red square must be covered, any valid tiling requires at least mn tiles.

Construction: It remains to show that a tiling with mn tiles exists for $m, n \geq 4$. First, notice that a $2 \times (2k-1)$ rectangle (with $k \geq 3$) can be tiled using exactly k tiles: one of Type 1, then $k-2$ of Type 2, and one of Type 1. We can decompose a general $(2m-1) \times (2n-1)$ rectangle into:

1. A 7×7 square in the corner (tiled by 16 tiles),
2. $m - 4$ rectangles of dimension 2×7 (tiled by 4 tiles each),
3. $n - 4$ rectangles of dimension $(2m-1) \times 2$ (tiled by m tiles each).

Summing the tile counts yields $16 + 4(m-4) + m(n-4) = mn$ tiles in total. □

Solution to Problem 5 (Putnam 2018, B1)

Proof. The vectors $v \in P$ satisfying the condition are those of the form

$$v = \begin{pmatrix} 1 \\ b \end{pmatrix} \quad \text{where } b \in \{0, 2, 4, \dots, 100\} \text{ is an even integer.}$$

Necessity: The set P contains $3 \times 101 = 303$ vectors. The total sum of all vectors in P is

$$\Sigma(P) = \sum_{a=0}^{2} \sum_{b=0}^{100} \begin{pmatrix} a \\ b \end{pmatrix} = 101\left(0+1+2\right)\mathbf{e}_1 + 3\left(\textstyle\sum_{b=0}^{100} b\right)\mathbf{e}_2 = \begin{pmatrix} 303 \\ 15150 \end{pmatrix}.$$

If $P \setminus \{v\}$ is partitioned into two sets S_1, S_2 of equal size (151 elements each) and equal sum $\Sigma(S_1) = \Sigma(S_2)$, then

$$\Sigma(P) = v + \Sigma(S_1) + \Sigma(S_2) = v + 2\Sigma(S_1).$$

Reducing this modulo 2:

$$v = \begin{pmatrix} a \\ b \end{pmatrix} \equiv \Sigma(P) = \begin{pmatrix} 303 \\ 15150 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \pmod 2.$$

Thus, a must be odd (so $a = 1$), and b must be even.

Sufficiency: Suppose $v = \begin{pmatrix} 1 \\ b \end{pmatrix}$ where b is even. We can pair up all elements $(x, y) \in P \setminus \{(1, 50)\}$ into 151 complementary pairs of the form (x, y) and $(2 - x, 100 - y)$, each summing to $(2, 100)$.

If $b \neq 50$, three of these pairs are:

$$\left\{ \begin{pmatrix} 1 \\ b \end{pmatrix}, \begin{pmatrix} 1 \\ 100 - b \end{pmatrix} \right\}, \quad \left\{ \begin{pmatrix} 2 \\ b \end{pmatrix}, \begin{pmatrix} 0 \\ 100 - b \end{pmatrix} \right\}, \quad \left\{ \begin{pmatrix} 2 \\ 25 + b/2 \end{pmatrix}, \begin{pmatrix} 0 \\ 75 - b/2 \end{pmatrix} \right\}.$$

We omit $v = (1, b)$, assign half of the remaining 148 pairs equally to S_1 and S_2, and distribute the remaining elements from the special pairs along with $(1, 50)$ such that both S_1 and S_2 receive equal sums and sizes. A similar explicit distribution works when $b = 50$. Hence all vectors of the form $(1, b)$ with b even are valid solutions. $\square$