Putnam Practice Session 5

Functional Equations: Substitution, Iteration, and Inequalities

Overview & Pedagogical Goals

This study guide and practice packet covers classical and competition-level techniques for solving functional equations. Unlike differential equations, functional equations express relationships between function values at different arguments without derivatives. Core methodologies include:

1. **Strategic Variable Substitutions**: Exploiting symmetry, involutions, or cyclic transformations to construct systems of linear algebraic equations.
2. **Cauchy-Type Functional Equations**: Recognizing standard additive, multiplicative, and logarithmic identities and their structural solutions.
3. **Iterated Maps and Boundary Limits**: Analyzing sequence iterations $x_{n+1} = T(x_n)$ and taking limits alongside continuity or monotonicity properties.
4. **Logarithmic Transformations and Inequalities**: Linearizing non-linear functional bounds to deduce power-law scaling.

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1 Study Guide & Core Principles

1.1 1. Strategic Substitutions and Algebraic Reduction

The most direct approach to solving a functional equation is substituting specific values or transformations into the variables to isolate $f(x)$.

Definition 1 (Cyclic Involutions). *A transformation $T(x)$ is called an* ***involution*** *if $T(T(x)) = x$, or cyclic of order k if $T^{(k)}(x) = x$. If a functional equation involves $f(x)$ and $f(T(x))$, replacing x with $T(x), T^{(2)}(x), \dots$ generates a system of linear equations in the unknowns $f(x), f(T(x)), \dots$.*

Example 1 (Order-2 System Isolation). *Find all functions $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ satisfying*

$$f(x) + 2f\left(\frac{1}{x}\right) = x \quad \textit{for all } x \neq 0.$$

Substitute $x \mapsto \frac{1}{x}$ into the equation:

$$f\left(\frac{1}{x}\right) + 2f(x) = \frac{1}{x}.$$

We now have a system of two linear equations in $f(x)$ and $f(1/x)$:

$$\begin{cases} f(x) + 2f(1/x) = x \\ 2f(x) + f(1/x) = 1/x \end{cases}$$

Multiplying the second equation by 2 and subtracting the first gives $3f(x) = \frac{2}{x} - x$, so $f(x) = \frac{2-x^2}{3x}$.

1.2 2. Cauchy's Functional Equation and Variations

Theorem 1 (Cauchy's Additive Equation). *Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If f is continuous (or bounded on an interval, or monotonic), then $f(x) = cx$ for some constant $c \in \mathbb{R}$.*

Related standard variants (assuming continuity):

- **Multiplicative**: $f(x+y) = f(x)f(y) \implies f(x) = a^x$ or $f(x) = 0$.
- **Logarithmic**: $f(xy) = f(x) + f(y) \implies f(x) = c \ln |x|$.
- **Power-law**: $f(xy) = f(x)f(y) \implies f(x) = x^c$ or $f(x) = 0$.

1.3 3. Iterated Function Maps and Invariance

When a functional equation relates $f(x)$ to $f(T(x))$, one can study the orbit $x_0, x_1 = T(x_0), x_2 = T(x_1), \ldots$.

Example 2 (Contraction Iteration). *Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies $f(x) = f(x/2)$ for all x. Iterating n times yields $f(x) = f(x/2^n)$. Taking $n \to \infty$, since $x/2^n \to 0$ and f is continuous, we get $f(x) = f(0)$ for all x. Thus, f must be a constant function.*

1.4 4. Logarithmic Transformations for Functional Inequalities

Functional inequalities of the form $(f(x))^a \leq f(y) \leq (f(x))^b$ for arguments $x^a \leq y \leq x^b$ can be linearized by taking logarithms twice: setting $g(t) = \ln f(e^t)$ transforms exponentiation into multiplication.

2 Selected Putnam Problems

Here are five classical Putnam problems from 2006 onward that illustrate these non-differential functional equation techniques.

Problem 1 (Putnam 2008, Problem A1)

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that

$$f(x, y) + f(y, z) + f(z, x) = 0$$

for all real numbers x, y, z. Prove that there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = g(x) - g(y)$ for all $x, y \in \mathbb{R}$.

Problem 2 (Putnam 2012, Problem A3)

Let $f : [-1, 1] \to \mathbb{R}$ be a continuous function such that $f(0) = 1$ and

$$f(x) = \frac{2 - x^2}{2} f\left(\frac{x^2}{2 - x^2}\right) \quad \text{for all } x \in [-1, 1].$$

Assuming that $\lim_{x \to 1^-} \frac{f(x)}{\sqrt{1-x}}$ exists and is finite, determine $f(x)$ explicitly.

Problem 3 (Putnam 2016, Problem A3)

Find the value of the integral

$$\int_0^1 f(x)\, dx$$

for the function $f : \mathbb{R} \setminus \{0, 1\} \to \mathbb{R}$ satisfying

$$f(x) + f\left(1 - \frac{1}{x}\right) = \arctan(x) \quad \text{for all } x \neq 0, 1.$$

Short Version Ends Here!

The “short” version of this document ends here, I will distribute it by email. The “full” version includes two more problems and full solutions. I’ll release it online during our meeting.