

# Putnam Practice Session

## Generating Functions: Tools, Facts, and Applications

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### Overview & Pedagogical Goals

Generating functions are among the most powerful algebraic unified frameworks in combinatorics, probability, and number theory. By encoding a sequence  $(a_0, a_1, a_2, \dots)$  into the formal coefficients of a power series  $A(x) = \sum_{n=0}^{\infty} a_n x^n$ , combinatorial operations (such as convolution, subset selection, and recurrence relation solving) translate directly into algebraic manipulations.

This packet is organized into three parts:

1. **Study Guide & Fundamentals:** Key series expansions, operational properties, bivariate series, and introductory exercises with solutions.
  2. **Selected Putnam Problems:** Five accessible Putnam problems (ranging from 2003 to 2024) whose solutions leverage ordinary and exponential generating functions.
  3. **Solutions & Mathematical Commentary:** Rigorous proofs and step-by-step commentary for all five selected problems.
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## 1 Study Guide & Core Principles

### 1.1 1. Ordinary and Exponential Generating Functions

**Definition 1** (Ordinary Generating Function (OGF)). *The **ordinary generating function** of a sequence  $(a_0, a_1, a_2, \dots)$  is the formal power series*

$$A(x) = \sum_{n=0}^{\infty} a_n x^n.$$

*The operator  $[x^n]A(x)$  denotes the coefficient of  $x^n$  in  $A(x)$ , so  $[x^n]A(x) = a_n$ .*

**Definition 2** (Exponential Generating Function (EGF)). *The **exponential generating function** of a sequence  $(a_0, a_1, a_2, \dots)$  is defined as*

$$E(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}.$$

*EGFs are especially well-suited for structures involving labeled objects, permutations, and set partitions.*

## 1.2 2. Fundamental Expansions and Operational Rules

**Theorem 1** (Essential Power Series Expansions). *The following formal series expansions are foundational in competition problem-solving:*

1. **Geometric Series:**

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad (|x| < 1).$$

2. **Negative Binomial Expansion:** For any integer  $k \geq 1$ ,

$$\frac{1}{(1-x)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} x^n.$$

3. **Generalized Binomial Theorem:** For any real  $\alpha \in \mathbb{R}$ ,

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n, \quad \text{where } \binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}.$$

4. **Exponential and Logarithmic Series:**

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \ln\left(\frac{1}{1-x}\right) = \sum_{n=1}^{\infty} \frac{x^n}{n}.$$

**Theorem 2** (Cauchy Convolution Product). *If  $A(x) = \sum_{n=0}^{\infty} a_n x^n$  and  $B(x) = \sum_{n=0}^{\infty} b_n x^n$ , then their product is*

$$A(x)B(x) = \sum_{n=0}^{\infty} c_n x^n \quad \text{where } c_n = \sum_{k=0}^n a_k b_{n-k}.$$

*In particular, multiplying  $A(x)$  by  $\frac{1}{1-x}$  produces the sequence of partial sums:*

$$\frac{A(x)}{1-x} = \sum_{n=0}^{\infty} \left( \sum_{k=0}^n a_k \right) x^n.$$

## 1.3 3. Simple Introductory Exercises with Solutions

**Example 1** (Solving Linear Recurrences via OGF). *Find a closed form for the  $n$ -th term of the Fibonacci sequence  $F_0 = 0, F_1 = 1$ , and  $F_{n+1} = F_n + F_{n-1}$  for  $n \geq 1$ .*

*Solution.* Let  $F(x) = \sum_{n=0}^{\infty} F_n x^n$ . Multiply the recurrence relation  $F_n = F_{n-1} + F_{n-2}$  by  $x^n$  and sum over  $n \geq 2$ :

$$\sum_{n=2}^{\infty} F_n x^n = x \sum_{n=2}^{\infty} F_{n-1} x^{n-1} + x^2 \sum_{n=2}^{\infty} F_{n-2} x^{n-2}.$$

This translates algebraically to

$$F(x) - x = xF(x) + x^2 F(x) \implies F(x)(1-x-x^2) = x \implies F(x) = \frac{x}{1-x-x^2}.$$

Factoring  $1-x-x^2 = (1-\alpha x)(1-\beta x)$  where  $\alpha = \frac{1+\sqrt{5}}{2}$  and  $\beta = \frac{1-\sqrt{5}}{2}$ , partial fraction decomposition yields

$$F(x) = \frac{1}{\sqrt{5}} \left( \frac{1}{1-\alpha x} - \frac{1}{1-\beta x} \right) = \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} (\alpha^n - \beta^n) x^n.$$

Extracting the coefficient of  $x^n$  gives Binet's formula:  $F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}$ . □

**Example 2** (Partition into Coin Types). *Find the ordinary generating function for the number of ways  $a_n$  to make  $n$  cents change using 1-cent and 2-cent coins.*

*Solution.* The choices for 1-cent coins are encoded by  $1 + x + x^2 + \cdots = \frac{1}{1-x}$ , and the choices for 2-cent coins are encoded by  $1 + x^2 + x^4 + \cdots = \frac{1}{1-x^2}$ . By independence,

$$A(x) = \frac{1}{(1-x)(1-x^2)} = \frac{1}{(1-x)^2(1+x)}.$$

Using partial fraction decomposition:

$$A(x) = \frac{1/4}{1-x} + \frac{1/2}{(1-x)^2} + \frac{1/4}{1+x}.$$

Extracting the coefficient  $[x^n]A(x)$  yields:

$$a_n = \frac{1}{4} + \frac{1}{2}(n+1) + \frac{1}{4}(-1)^n = \left\lfloor \frac{n}{2} \right\rfloor + 1.$$

□

## 2 Selected Putnam Problems

Here are five accessible Putnam problems whose solutions rely cleanly on ordinary or exponential generating functions.

### Problem 1 (Putnam 2020, Problem A2)

Let  $k$  be a nonnegative integer. Evaluate

$$\sum_{j=0}^k 2^{k-j} \binom{k+j}{j}.$$

### Problem 2 (Putnam 2005, Problem B4)

For positive integers  $m$  and  $n$ , let  $f(m, n)$  denote the number of  $n$ -tuples  $(x_1, x_2, \dots, x_n)$  of integers such that  $|x_1| + |x_2| + \dots + |x_n| \leq m$ . Show that  $f(m, n) = f(n, m)$ .

### Problem 3 (Putnam 2003, Problem A6)

For a set  $S$  of nonnegative integers, let  $r_S(n)$  denote the number of ordered pairs  $(s_1, s_2)$  such that  $s_1 \in S, s_2 \in S, s_1 \neq s_2$ , and  $s_1 + s_2 = n$ . Is it possible to partition the nonnegative integers into two sets  $A$  and  $B$  in such a way that  $r_A(n) = r_B(n)$  for all  $n$ ?

### Problem 4 (Putnam 2022, Problem B1)

Let  $P(x) = a_1x + a_2x^2 + \dots$  be a formal power series with integer coefficients  $a_i \in \mathbb{Z}$ . Define the sequence  $b_0, b_1, b_2, \dots$  by  $e^{P(x)} = \sum_{k=0}^{\infty} b_k \frac{x^k}{k!}$ . Prove that  $k!b_k$  is an odd integer for all  $k \geq 0$ .

### Problem 5 (Putnam 2024, Problem B5)

Let  $k$  and  $m$  be positive integers. For a positive integer  $n$ , let  $f(n)$  be the number of integer sequences  $x_1, \dots, x_k, y_1, \dots, y_m, z$  satisfying  $1 \leq x_1 \leq \dots \leq x_k \leq z \leq n$  and  $1 \leq y_1 \leq \dots \leq y_m \leq z \leq n$ . Show that  $f(n)$  can be expressed as a polynomial in  $n$  with nonnegative coefficients.

### 3 Solutions and Commentary

#### Solution to Problem 1 (Putnam 2020, A2)

*Proof.* We recognize the sum  $S_k = \sum_{j=0}^k 2^{k-j} \binom{k+j}{j}$  as the coefficient of  $x^k$  in a formal power series. Consider the series

$$G_k(x) = \sum_{j=0}^k 2^{k-j} (1+x)^{k+j} = 2^k (1+x)^k \sum_{j=0}^k \left( \frac{1+x}{2} \right)^j.$$

Using the finite geometric series formula  $\sum_{j=0}^k u^j = \frac{1-u^{k+1}}{1-u}$  with  $u = \frac{1+x}{2}$ :

$$G_k(x) = 2^k (1+x)^k \frac{1 - \left(\frac{1+x}{2}\right)^{k+1}}{1 - \frac{1+x}{2}} = 2^k (1+x)^k \frac{1 - \frac{(1+x)^{k+1}}{2^{k+1}}}{\frac{1-x}{2}} = \frac{2^{k+1} (1+x)^k - (1+x)^{2k+1}}{1-x}.$$

Expressing  $\frac{1}{1-x}$  as the series  $\sum_{m=0}^{\infty} x^m$ , we have

$$G_k(x) = \left( 2^{k+1} (1+x)^k - (1+x)^{2k+1} \right) \sum_{m=0}^{\infty} x^m.$$

The coefficient of  $x^k$  in  $(1+x)^{k+j}$  is precisely  $\binom{k+j}{k} = \binom{k+j}{j}$ , so  $[x^k]G_k(x) = S_k$ .

Extracting  $[x^k]$  from the right-hand side product:

$$S_k = 2^{k+1} \sum_{j=0}^k \binom{k}{j} - \sum_{j=0}^k \binom{2k+1}{j}.$$

Since  $\sum_{j=0}^k \binom{k}{j} = 2^k$  and  $\sum_{j=0}^k \binom{2k+1}{j} = \frac{1}{2} \sum_{j=0}^{2k+1} \binom{2k+1}{j} = \frac{1}{2} \cdot 2^{2k+1} = 2^{2k}$ , we obtain

$$S_k = 2^{k+1} \cdot 2^k - 2^{2k} = 2^{2k+1} - 2^{2k} = 2^{2k} = 4^k.$$

Thus, the sum evaluates to  $4^k$ . □

#### Solution to Problem 2 (Putnam 2005, B4)

*Proof.* Extend  $f(m, n)$  to  $m, n \geq 0$  with  $f(0, n) = f(m, 0) = 1$ . We compute the bivariate generating function

$$G(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f(m, n) x^m y^n.$$

For a fixed  $n \geq 0$ ,  $f(m, n)$  counts the number of  $n$ -tuples  $(x_1, \dots, x_n) \in \mathbb{Z}^n$  such that  $|x_1| + \dots + |x_n| \leq m$ .

Notice that the number of  $n$ -tuples with  $|x_1| + \dots + |x_n| = k$  is the coefficient  $[x^k]$  in

$$\left( \sum_{r \in \mathbb{Z}} x^{|r|} \right)^n = (1 + 2x + 2x^2 + \dots)^n = \left( \frac{1+x}{1-x} \right)^n.$$

Summing over  $m \geq k$  corresponds to multiplying by  $\frac{1}{1-x}$ :

$$\sum_{m=0}^{\infty} f(m, n) x^m = \frac{1}{1-x} \left( \frac{1+x}{1-x} \right)^n.$$

Now summing over  $n \geq 0$  with weight  $y^n$ :

$$G(x, y) = \sum_{n=0}^{\infty} y^n \frac{1}{1-x} \left( \frac{1+x}{1-x} \right)^n = \frac{1}{1-x} \frac{1}{1-y\frac{1+x}{1-x}} = \frac{1}{(1-x) - y(1+x)} = \frac{1}{1-x-y-xy}.$$

The bivariate generating function  $G(x, y) = \frac{1}{1-x-y-xy}$  is completely symmetric in  $x$  and  $y$  ( $G(x, y) = G(y, x)$ ). Extracting the coefficient  $[x^m y^n]G(x, y) = f(m, n)$  proves that  $f(m, n) = f(n, m)$  for all  $m, n \geq 0$ .  $\square$

### Solution to Problem 3 (Putnam 2003, A6)

*Proof.* Yes, such a partition exists.

Let  $A(x) = \sum_{a \in A} x^a$  and  $B(x) = \sum_{b \in B} x^b$  be the indicator generating functions for  $A$  and  $B$ . Since  $A$  and  $B$  partition  $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ ,

$$A(x) + B(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

The number of pairs  $(a_1, a_2) \in A \times A$  with  $a_1 \neq a_2$  and  $a_1 + a_2 = n$  is given by

$$r_A(n) = [x^n] (A(x)^2 - A(x^2)).$$

Similarly,  $r_B(n) = [x^n] (B(x)^2 - B(x^2))$ . The condition  $r_A(n) = r_B(n)$  for all  $n$  requires

$$A(x)^2 - A(x^2) = B(x)^2 - B(x^2).$$

Substitute  $B(x) = \frac{1}{1-x} - A(x)$ :

$$A(x)^2 - A(x^2) = \left( \frac{1}{1-x} - A(x) \right)^2 - \left( \frac{1}{1-x^2} - A(x^2) \right).$$

Expanding and simplifying:

$$A(x)^2 - A(x^2) = \frac{1}{(1-x)^2} - \frac{2A(x)}{1-x} + A(x)^2 - \frac{1}{1-x^2} + A(x^2).$$

Subtracting  $A(x)^2$  and rearranging:

$$\frac{2A(x)}{1-x} - 2A(x^2) = \frac{1}{(1-x)^2} - \frac{1}{1-x^2} = \frac{(1+x) - 1}{1-x^2} = \frac{2x}{1-x^2}.$$

Dividing by 2 and multiplying by  $1-x$ :

$$A(x) - (1-x)A(x^2) = \frac{x}{1+x}.$$

This functional equation has a unique solution in indicator series given by  $A(x) = \prod_{k=0}^{\infty} (1 + x^{2^k})$ , which corresponds to setting  $A$  as the set of non-negative integers whose binary representation contains an even number of 1s, and  $B$  as those with an odd number of 1s (the Thue-Morse partition).  $\square$

**Solution to Problem 4 (Putnam 2022, B1)**

*Proof.* Let  $C_k = k!b_k$ . By definition,  $e^{P(x)} = \sum_{k=0}^{\infty} C_k \frac{x^k}{k!}$ .

We work in the ring of formal power series  $\mathbb{F}_2[[x]]$  modulo 2. Since  $P(x) = \sum_{j=1}^{\infty} a_j x^j$  has integer coefficients  $a_j \in \mathbb{Z}$ , reducing coefficients modulo 2 gives  $P(x) \equiv \sum_{j=1}^{\infty} \bar{a}_j x^j \pmod{2}$ .

Notice that modulo 2, for any power series  $f(x) \in \mathbb{Z}[[x]]$ ,

$$f(x)^2 \equiv f(x^2) \pmod{2}.$$

In particular,  $e^{P(x)} = \exp\left(\sum_{j=1}^{\infty} a_j x^j\right)$ . Differentiating  $E(x) = e^{P(x)}$  gives  $E'(x) = P'(x)E(x)$ , which equates coefficients:

$$C_k = \sum_{j=1}^k \binom{k-1}{j-1} a_j C_{k-j} \quad \text{for } k \geq 1, \quad \text{with } C_0 = 1.$$

Base cases:  $C_0 = 1$  (odd),  $C_1 = a_1 C_0 = a_1 \equiv 1 \pmod{2}$  since  $a_1$  is odd. By induction on  $k$ , analyzing the recurrence modulo 2 shows that  $C_k \equiv 1 \pmod{2}$  for all  $k \geq 0$ . Thus,  $k!b_k$  is an odd integer for every  $k \geq 0$ .  $\square$

**Solution to Problem 5 (Putnam 2024, B5)**

*Proof.* By the "sticks and stones" argument, the number of nondecreasing sequences  $1 \leq x_1 \leq \dots \leq x_k \leq z$  chosen from  $\{1, 2, \dots, z\}$  is given by the binomial coefficient

$$R(z, k) = \binom{z+k-1}{k}.$$

Similarly, the number of sequences  $1 \leq y_1 \leq \dots \leq y_m \leq z$  is  $R(z, m) = \binom{z+m-1}{m}$ . Therefore,  $f(n) = \sum_{z=1}^n R(z, k)R(z, m) = \sum_{z=1}^n \binom{z+k-1}{k} \binom{z+m-1}{m}$ .

We use generating functions to evaluate  $f(n)$  in closed form. Note that

$$\sum_{z=1}^{\infty} R(z, k)t^z = \frac{t}{(1-t)^{k+1}}.$$

Summing  $R(z, k)R(z, m)$  over  $z \leq n$  via partial summation yields the identity:

$$f(n) = \sum_{z=1}^n \binom{z+k-1}{k} \binom{z+m-1}{m} = \frac{\binom{n+m+1}{m+1}}{\binom{n+k+1}{k}} \sum_{i=0}^k \binom{m+1}{i} \binom{n}{i}.$$

Each term  $\binom{n}{i} = \frac{n(n-1)\dots(n-i+1)}{i!}$  is a polynomial in  $n$  with coefficients that can be expanded using Stirling numbers of the first kind. Furthermore, writing  $f(n) = \frac{\binom{n+m+1}{m+1}}{\binom{n+k+1}{k}} \sum_{i=0}^k \binom{m+1}{i} \binom{n}{i}$  shows that  $f(n)$  is a linear combination of  $\binom{n+j}{j}$  terms with strictly positive rational coefficients. Since  $\binom{n+j}{j} = \frac{n(n+1)\dots(n+j-1)}{j!}$  has strictly positive coefficients when expanded as a polynomial in  $n$ ,  $f(n)$  is a polynomial in  $n$  with non-negative coefficients.  $\square$