

Putnam Practice Session 7

Classical Inequalities and Bounding Techniques

1 Study Guide & Core Principles

For an in-depth treatment with extensive historical examples, readers should consult *Putnam and Beyond* (Section 2.1: *Identities and Inequalities*).

1.1 1. Non-negativity of Squares ($x^2 \geq 0$) and Completing the Square

The most fundamental algebraic inequality states that $x^2 \geq 0$ for all $x \in \mathbb{R}$, with equality if and only if $x = 0$. In competition problems, this principle manifests in several ways:

- **Completing the Square:** Rewriting quadratic expressions $ax^2 + bx + c = a(x + \frac{b}{2a})^2 + (c - \frac{b^2}{4a})$.
- **Lagrange's Identity:** Sums of pairwise squared differences, such as

$$a^2 + b^2 + c^2 - ab - bc - ca = \frac{1}{2} [(a-b)^2 + (b-c)^2 + (c-a)^2] \geq 0.$$

- **Integral Completing the Square:** Factoring integrands $A(x)f(x)^2 + B(x)f(x) + C(x)$ into $A(x)(f(x) - g(x))^2 + H(x)$ to establish sharp global integral bounds.

1.2 2. The AM-GM Inequality and Variations

Theorem 1 (AM-GM Inequality). *For any non-negative real numbers $x_1, x_2, \dots, x_n$,*

$$\frac{x_1 + x_2 + \dots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \dots x_n},$$

with equality if and only if $x_1 = x_2 = \dots = x_n$.

Theorem 2 (Weighted AM-GM Inequality). *If $w_1, w_2, \dots, w_n > 0$ with $\sum_{i=1}^n w_i = 1$, then for any non-negative x_i ,*

$$\sum_{i=1}^n w_i x_i \geq \prod_{i=1}^n x_i^{w_i}.$$

Example 1 (Strategic Weighting). *A common tactic when evaluating sums like $\sum \frac{n}{2^n} (a_1 \dots a_n)^{1/n}$ is choosing scaling multipliers $c_k = 4^k$ so that applying AM-GM to the terms $c_k a_k$ allows the inner products to be uncoupled and summed via geometric series.*

1.3 3. Cauchy–Schwarz and Hölder Inequalities

Theorem 3 (Cauchy–Schwarz Inequality). *For real sequences $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$,*

$$\left(\sum_{i=1}^n a_i b_i \right)^2 \leq \left(\sum_{i=1}^n a_i^2 \right) \left(\sum_{i=1}^n b_i^2 \right).$$

In integral form, for real-valued square-integrable functions f, g on $[a, b]$,

$$\left(\int_a^b f(x)g(x) \, dx \right)^2 \leq \left(\int_a^b f(x)^2 \, dx \right) \left(\int_a^b g(x)^2 \, dx \right).$$

Theorem 4 (Hölder's Inequality). If $p, q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\int_a^b |f(x)g(x)| \, dx \leq \left(\int_a^b |f(x)|^p \, dx \right)^{1/p} \left(\int_a^b |g(x)|^q \, dx \right)^{1/q}.$$

1.4 4. Convexity and Jensen's Inequality

Definition 1 (Convex Function). A real-valued function f on an interval I is **convex** if $f''(x) \geq 0$ for all $x \in I$.

Theorem 5 (Jensen's Inequality). If f is convex on I , $\lambda_1, \dots, \lambda_n \geq 0$ with $\sum \lambda_i = 1$, and $x_i \in I$, then

$$f\left(\sum_{i=1}^n \lambda_i x_i\right) \leq \sum_{i=1}^n \lambda_i f(x_i).$$

2 Selected Putnam Problems

Here are five selected Putnam problems involving inequalities, arranged in increasing order of difficulty.

Problem 1 (Putnam 2003, Problem A2)

Let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be nonnegative real numbers. Show that

$$(a_1 a_2 \cdots a_n)^{1/n} + (b_1 b_2 \cdots b_n)^{1/n} \leq [(a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n)]^{1/n}.$$

Problem 2 (Putnam 2004, Problem B2)

Let m and n be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m} \frac{n!}{n^n}.$$

Problem 3 (Putnam 2006, Problem B5)

For any continuous real-valued function $f : [0, 1] \rightarrow \mathbb{R}$, evaluate

$$\max_f \left(\int_0^1 x^2 f(x) dx - \int_0^1 x (f(x))^2 dx \right).$$

Problem 4 (Putnam 2017, Problem A3)

Let a and b be real numbers with $a < b$, and let f and g be continuous functions from $[a, b]$ to $(0, \infty)$ such that $\int_a^b f(x) dx = \int_a^b g(x) dx$ but $f \neq g$. For every positive integer n , define

$$I_n = \int_a^b \frac{(f(x))^{n+1}}{(g(x))^n} dx.$$

Show that $I_1, I_2, I_3, \dots$ is an increasing sequence with $\lim_{n \rightarrow \infty} I_n = \infty$.

Problem 5 (Putnam 2021, Problem B2)

Determine the maximum value of the sum

$$S = \sum_{n=1}^{\infty} \frac{n}{2^n} (a_1 a_2 \cdots a_n)^{1/n}$$

over all sequences $a_1, a_2, a_3, \dots$ of nonnegative real numbers satisfying $\sum_{k=1}^{\infty} a_k = 1$.

3 Solutions and Commentary

Solution to Problem 1 (Putnam 2003, A2)

Proof. Assume without loss of generality that $a_i + b_i > 0$ for each $i \in \{1, \dots, n\}$ (otherwise both sides of the inequality evaluate to 0).

Applying the AM-GM inequality to the terms $\frac{a_i}{a_i + b_i}$ gives:

$$\left(\frac{a_1 a_2 \cdots a_n}{(a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n)} \right)^{1/n} \leq \frac{1}{n} \sum_{i=1}^n \frac{a_i}{a_i + b_i}.$$

Similarly, applying the AM-GM inequality with a and b reversed yields:

$$\left(\frac{b_1 b_2 \cdots b_n}{(a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n)} \right)^{1/n} \leq \frac{1}{n} \sum_{i=1}^n \frac{b_i}{a_i + b_i}.$$

Adding these two inequalities together:

$$\frac{(a_1 a_2 \cdots a_n)^{1/n} + (b_1 b_2 \cdots b_n)^{1/n}}{[(a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n)]^{1/n}} \leq \frac{1}{n} \sum_{i=1}^n \frac{a_i + b_i}{a_i + b_i} = \frac{1}{n} \cdot n = 1.$$

Clearing the denominator proves the desired inequality. □

Solution to Problem 2 (Putnam 2004, B2)

Proof. The desired inequality can be rewritten in equivalent form as:

$$(m+n)^{m+n} > \binom{m+n}{m} m^m n^n.$$

Consider the binomial expansion of $(m+n)^{m+n}$:

$$(m+n)^{m+n} = \sum_{k=0}^{m+n} \binom{m+n}{k} m^k n^{m+n-k}.$$

Setting $k = m$ in the summation yields the single term $\binom{m+n}{m} m^m n^n$.

Since $m, n \geq 1$, the expansion contains this term as well as several other strictly positive terms (e.g., for $k = 0$ and $k = m+n$). Therefore:

$$(m+n)^{m+n} > \binom{m+n}{m} m^m n^n = \frac{(m+n)!}{m! n!} m^m n^n.$$

Rearranging terms yields $\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m} \frac{n!}{n^n}$, as claimed. □

Solution to Problem 3 (Putnam 2006, B5)

Proof. We complete the square inside the integrand with respect to $f(x)$:

$$x^2 f(x) - x(f(x))^2 = \frac{x^3}{4} - x \left(f(x) - \frac{x}{2} \right)^2.$$

Integrating this identity over $[0, 1]$ gives:

$$\int_0^1 x^2 f(x) dx - \int_0^1 x(f(x))^2 dx = \int_0^1 \frac{x^3}{4} dx - \int_0^1 x \left(f(x) - \frac{x}{2}\right)^2 dx.$$

Since $x \geq 0$ for all $x \in [0, 1]$ and $\left(f(x) - \frac{x}{2}\right)^2 \geq 0$, the second integral is non-negative. Hence:

$$\int_0^1 x^2 f(x) dx - \int_0^1 x(f(x))^2 dx \leq \int_0^1 \frac{x^3}{4} dx = \left[\frac{x^4}{16}\right]_0^1 = \frac{1}{16}.$$

Equality is uniquely attained for the continuous function $f(x) = \frac{x}{2}$. Thus, the maximum value is $\frac{1}{16}$. $\square$

Solution to Problem 4 (Putnam 2017, A3)

Proof. We extend the definition of I_n to $n = 0$:

$$I_0 = \int_a^b f(x) dx = \int_a^b g(x) dx > 0.$$

We first prove $I_1 > I_0$. Notice that:

$$I_1 - I_0 = \int_a^b \left(\frac{(f(x))^2}{g(x)} - f(x) \right) dx = \int_a^b \frac{f(x)}{g(x)} (f(x) - g(x)) dx.$$

Since $\int_a^b (f(x) - g(x)) dx = 0$, we can subtract $g(x) - f(x)$ inside the integral:

$$I_1 - I_0 = \int_a^b \frac{(f(x) - g(x))^2}{g(x)} dx.$$

The integrand $\frac{(f(x)-g(x))^2}{g(x)}$ is continuous, non-negative, and not identically zero (since $f \neq g$). Thus $I_1 - I_0 > 0$, so $I_1 > I_0$.

Next, for any $n \geq 0$, applying the Cauchy–Schwarz inequality for integrals to $f(x)^{(n+1)/2}/g(x)^{n/2}$ and $f(x)^{(n+3)/2}/g(x)^{(n+2)/2}$ yields:

$$I_{n+1}^2 = \left(\int_a^b \frac{(f(x))^{n+2}}{(g(x))^{n+1}} dx \right)^2 \leq \left(\int_a^b \frac{(f(x))^{n+1}}{(g(x))^n} dx \right) \left(\int_a^b \frac{(f(x))^{n+3}}{(g(x))^{n+2}} dx \right) = I_n I_{n+2}.$$

Therefore, the ratio sequence $\frac{I_{n+1}}{I_n}$ is non-decreasing:

$$\frac{I_{n+1}}{I_n} \geq \frac{I_1}{I_0} > 1 \quad \text{for all } n \geq 0.$$

This implies $I_{n+1} > I_n$ for all $n \geq 1$, so I_n is strictly increasing.

Finally, express I_n/I_0 as a product of ratios:

$$\frac{I_n}{I_0} = \prod_{k=0}^{n-1} \frac{I_{k+1}}{I_k} \geq \left(\frac{I_1}{I_0} \right)^n.$$

Since $I_1/I_0 > 1$, taking $n \rightarrow \infty$ gives $\lim_{n \rightarrow \infty} I_n = \infty$. $\square$

Solution to Problem 5 (Putnam 2021, B2)

Proof. The maximum value of the sum S is $\frac{2}{3}$.

Apply the AM-GM inequality to the n positive terms $4^1 a_1, 4^2 a_2, \dots, 4^n a_n$:

$$(4^1 a_1 \cdot 4^2 a_2 \cdots 4^n a_n)^{1/n} \leq \frac{1}{n} \sum_{k=1}^n 4^k a_k.$$

Note that $4^1 \cdot 4^2 \cdots 4^n = 4^{n(n+1)/2} = 2^{n(n+1)} = (2^{n+1})^n$. Taking the n -th root gives:

$$2^{n+1} (a_1 a_2 \cdots a_n)^{1/n} \leq \frac{1}{n} \sum_{k=1}^n 4^k a_k \implies \frac{n}{2^n} (a_1 a_2 \cdots a_n)^{1/n} \leq \frac{1}{2^{2n+1}} \sum_{k=1}^n 4^k a_k.$$

Summing over all $n \geq 1$ and interchanging the order of summation:

$$S = \sum_{n=1}^{\infty} \frac{n}{2^n} (a_1 \cdots a_n)^{1/n} \leq \sum_{n=1}^{\infty} \frac{1}{2^{2n+1}} \sum_{k=1}^n 4^k a_k = \frac{1}{2} \sum_{k=1}^{\infty} 4^k a_k \sum_{n=k}^{\infty} 4^{-n}.$$

The inner geometric sum evaluates to:

$$\sum_{n=k}^{\infty} 4^{-n} = \frac{4^{-k}}{1 - 1/4} = \frac{4}{3} \cdot 4^{-k}.$$

Substituting this back yields:

$$S \leq \frac{1}{2} \sum_{k=1}^{\infty} 4^k a_k \left(\frac{4}{3} \cdot 4^{-k} \right) = \frac{2}{3} \sum_{k=1}^{\infty} a_k = \frac{2}{3} (1) = \frac{2}{3}.$$

Equality holds in the AM-GM inequality when $4^1 a_1 = 4^2 a_2 = \cdots = 4^k a_k = \cdots = C$ for some constant C . Choosing $a_k = \frac{3}{4^k}$ yields $\sum_{k=1}^{\infty} a_k = 3 \cdot \frac{1/4}{1-1/4} = 1$, and $(a_1 \cdots a_n)^{1/n} = 3 \cdot 2^{-(n+1)}$. Substituting this sequence gives:

$$S = \sum_{n=1}^{\infty} \frac{n}{2^n} \cdot 3 \cdot 2^{-(n+1)} = \frac{3}{2} \sum_{n=1}^{\infty} \frac{n}{4^n} = \frac{3}{2} \cdot \frac{4}{9} = \frac{2}{3}.$$

Thus, the maximum value is $\frac{2}{3}$. □