

Strategic Weighting Trick for Putnam Problem 2021 B2

Overview

When applying the Arithmetic Mean–Geometric Mean (AM–GM) inequality to infinite series involving geometric means of products $(a_1 a_2 \cdots a_n)^{1/n}$, a direct application of standard AM–GM often fails because it produces bounds that are too loose and does not cancel the polynomial weights (like n).

This handout explains the **Strategic Weighting Trick** (also known as the *Carleman Weighting Method*) used to solve Putnam 2021, Problem B2.

1 1. The Failure of Direct AM–GM

Consider the objective sum in Putnam 2021 B2:

$$S = \sum_{n=1}^{\infty} \frac{n}{2^n} (a_1 a_2 \cdots a_n)^{1/n} \quad \text{subject to} \quad \sum_{k=1}^{\infty} a_k = 1.$$

If we apply standard unweighted AM–GM directly to $(a_1 \cdots a_n)^{1/n}$:

$$(a_1 a_2 \cdots a_n)^{1/n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Multiplying both sides by $\frac{n}{2^n}$ gives:

$$\frac{n}{2^n} (a_1 a_2 \cdots a_n)^{1/n} \leq \frac{a_1 + a_2 + \cdots + a_n}{2^n}.$$

Summing over $n \geq 1$ and swapping the order of summation yields:

$$S \leq \sum_{k=1}^{\infty} a_k \sum_{n=k}^{\infty} \frac{1}{2^n} = \sum_{k=1}^{\infty} \frac{a_k}{2^{k-1}}.$$

While this gives a valid upper bound, it is **too loose**:

1. It yields a maximum of 1 (attained at $a_1 = 1$), whereas the true maximum is $\frac{2}{3}$.
2. Equality in direct AM–GM requires $a_1 = a_2 = \cdots = a_n$, which contradicts $\sum a_k = 1$ for non-zero sequences.

2 2. Introducing Scaling Weights c_k

To sharpen the bound, we introduce a sequence of positive scaling multipliers $c_1, c_2, \dots, c_n > 0$. We multiply and divide inside the geometric mean:

$$(a_1 a_2 \cdots a_n)^{1/n} = \frac{1}{(c_1 c_2 \cdots c_n)^{1/n}} \cdot (c_1 a_1 \cdot c_2 a_2 \cdots c_n a_n)^{1/n}.$$

Applying standard AM–GM to the scaled terms $(c_1 a_1, c_2 a_2, \dots, c_n a_n)$ gives:

$$(c_1 a_1 \cdot c_2 a_2 \cdots c_n a_n)^{1/n} \leq \frac{1}{n} \sum_{k=1}^n c_k a_k.$$

Substituting this back yields the general weighted AM–GM bound:

$$(a_1 a_2 \cdots a_n)^{1/n} \leq \frac{1}{n \cdot (c_1 c_2 \cdots c_n)^{1/n}} \sum_{k=1}^n c_k a_k.$$

3 3. Why the Choice $c_k = 4^k$ Works

To determine the optimal choice for c_k , we observe two structural requirements:

1. **Canceling the n factor:** The $1/n$ factor from AM–GM must combine cleanly with $(c_1 \cdots c_n)^{1/n}$.
2. **Matching the 2^n denominator:** We need $(c_1 c_2 \cdots c_n)^{1/n}$ to simplify to 2^{n+1} .

Choosing $c_k = 4^k$:

- **Product of weights:**

$$c_1 c_2 \cdots c_n = 4^1 \cdot 4^2 \cdots 4^n = 4^{\frac{n(n+1)}{2}} = \left(4^{\frac{n+1}{2}}\right)^n = (2^{n+1})^n.$$

- **n -th root:**

$$(c_1 c_2 \cdots c_n)^{1/n} = 2^{n+1}.$$

Substituting $c_k = 4^k$ and $(c_1 \cdots c_n)^{1/n} = 2^{n+1}$ back into the weighted bound:

$$(a_1 a_2 \cdots a_n)^{1/n} \leq \frac{1}{n \cdot 2^{n+1}} \sum_{k=1}^n 4^k a_k.$$

4 4. Uncoupling and Summing the Series

Multiplying by $\frac{n}{2^n}$, notice how **the factor of n in the numerator cancels out completely**:

$$\frac{n}{2^n} (a_1 \cdots a_n)^{1/n} \leq \frac{n}{2^n} \cdot \frac{1}{n \cdot 2^{n+1}} \sum_{k=1}^n 4^k a_k = \frac{1}{2^{2n+1}} \sum_{k=1}^n 4^k a_k = \frac{1}{2 \cdot 4^n} \sum_{k=1}^n 4^k a_k.$$

Summing over $n \geq 1$ and interchanging the order of summation ($n \geq k$):

$$S = \sum_{n=1}^{\infty} \frac{n}{2^n} (a_1 \cdots a_n)^{1/n} \leq \frac{1}{2} \sum_{k=1}^{\infty} 4^k a_k \sum_{n=k}^{\infty} \frac{1}{4^n}.$$

The inner geometric sum is:

$$\sum_{n=k}^{\infty} 4^{-n} = \frac{4^{-k}}{1 - 1/4} = \frac{4}{3} \cdot 4^{-k}.$$

Substituting this back gives the tight upper bound:

$$S \leq \frac{1}{2} \sum_{k=1}^{\infty} 4^k a_k \left(\frac{4}{3} \cdot 4^{-k} \right) = \frac{2}{3} \sum_{k=1}^{\infty} a_k = \frac{2}{3} (1) = \frac{2}{3}.$$

5 5. Equality and Extremal Sequence

Equality holds in AM–GM when all scaled terms are equal:

$$4^1 a_1 = 4^2 a_2 = \cdots = 4^k a_k = \cdots = C \implies a_k = \frac{C}{4^k}.$$

Since $\sum a_k = 1$, we find C :

$$C \sum_{k=1}^{\infty} \frac{1}{4^k} = C \left(\frac{1/4}{1 - 1/4} \right) = \frac{C}{3} = 1 \implies C = 3.$$

Thus, the sequence $a_k = \frac{3}{4^k}$ satisfies $\sum a_k = 1$ and achieves the maximum bound of $\frac{2}{3}$ exactly.